

GROTHENDIECK'S LOCALIZATION PROBLEM

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ABSTRACT. The singularity of a fiber of a flat homomorphism of noetherian rings $\varphi: R \rightarrow S$ at a prime ideal $\mathfrak{p} \in \text{Spec } R$ is controlled by the singularity of the fiber of φ at any specialization of \mathfrak{p} and by the singularities of the formal fibers of R at \mathfrak{p} .

INTRODUCTION

Let $\varphi: R \rightarrow S$ be a homomorphism of commutative rings. For a prime ideal $\mathfrak{p} \in \text{Spec } R$, the residue field of the localization of R at \mathfrak{p} is denoted $k(\mathfrak{p})$, and the ring $k(\mathfrak{p}) \otimes_R S$ is called the *fiber* of φ at \mathfrak{p} . The fibers of the canonical maps from R to its completions in the \mathfrak{p} -adic topologies are known as the *formal fibers* of R .

For various properties \mathcal{P} of commutative rings, Grothendieck [9, (7.5)] considers the following Localization Problem: Suppose that φ is a flat homomorphism of noetherian rings and that the formal fibers of R have \mathcal{P} ; if the fibers of φ at the primes of R contracted from maximal ideals of S have \mathcal{P} , is it then true that all the fibers of φ have \mathcal{P} ? We obtain positive answers for properties related to “complete intersection,” “Gorenstein,” and “Cohen–Macaulay” in Section 4 and compare them to earlier ones in Section 5. However, the main thrust of this paper is to investigate the more general thesis formulated in the Abstract.

To make precise statements we describe some invariants which measure the complexity of the singularity of a local ring S with maximal ideal \mathfrak{n} and residue field ℓ . The formula $\text{cid } S = \text{rank}_\ell \mathbf{H}_1(\mathbf{K}) - \text{rank}_\ell \mathfrak{n}/\mathfrak{n}^2 + \dim S$, where \mathbf{K} denotes the Koszul complex on a minimal set of generators of \mathfrak{n} , defines the *complete intersection defect* of S (introduced under the name *deviation* and denoted $d(S)$ in [12]). This integer is non-negative, and vanishes if and only if the \mathfrak{n} -adic completion \widehat{S} is the quotient of a regular local ring by a regular sequence, that is, if S is a *complete intersection*, cf. also (1.2.a) below.

The *Cohen–Macaulay defect* of S is the non-negative integer $\text{cmd } S = \dim S - \text{depth } S$ (in [8, (16.4.9)] it is named *codepth* and denoted $\text{coprof } S$). An equality $\text{cmd } S = 0$ characterizes the *Cohen–Macaulay rings* S . In such S any two ideals generated by systems of parameters require the same number of irreducible ideals for their irredundant primary decom-

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positions, [16, Theorem 3], cf. also (1.2.c). This integer is called the *type* of S and denoted $\text{type } S$. *Gorenstein* rings may be described as the Cohen–Macaulay rings of type 1.

Each of the invariants introduced above is extended to arbitrary noetherian rings A by taking the supremum of its values on $A_{\mathfrak{p}}$, when \mathfrak{p} ranges over the maximal ideals of A , or – equivalently, cf. (1.4) – over its prime ideals.

Now we can state our main result, to be proved in Section 3, based on the existence [6] of Cohen factorizations for local homomorphisms to complete local rings. The necessary preparation is carried out in Section 2.

Main Theorem. *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of local rings.*

For any prime ideal \mathfrak{q} in S and its inverse image $\mathfrak{p} = \mathfrak{q} \cap R$ in R there are inequalities:

$$(a) \quad \text{cid } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cid}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{cid } S/\mathfrak{m}S + \text{cid}(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

$$(b) \quad \text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{cmd } S/\mathfrak{m}S + \text{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

(c) *If $S/\mathfrak{m}S$ and the formal fibers of R at \mathfrak{p} are Cohen–Macaulay, then $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ and the formal fibers of S at \mathfrak{q} are Cohen–Macaulay, and there is an inequality:*

$$\text{type}(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) \cdot \text{type}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{type}(S/\mathfrak{m}S) \cdot \text{type}(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

The inequalities of the Main Theorem are themselves limit cases of more general results. Suitably modified, the thesis stated in the Abstract is valid for many properties of *homomorphisms of finite flat dimension*, cf. [5] for extensions of (b) and (c). In fact, it is consideration of this situation which led us to the present point of view, but the proof of the Main Theorem in the flat case is technically less involved than that of the general result. It is also simpler than earlier proofs of Localization Theorems for \mathcal{P} = “complete intersection” or “Gorenstein,” and provides the first such theorem working in full generality for \mathcal{P} = “Cohen–Macaulay.”

1. DEFECTS AND TYPE

In this section (S, \mathfrak{n}, ℓ) denotes a local ring. All the results below are known, but some arguments are sketched so as to make the text independent of the theory of dualizing complexes and canonical modules.

(1.1) *Completions.* Complete intersection defect, Cohen–Macaulay defect, and type of Cohen–Macaulay rings are invariant under maximal-ideal-adic completions of local rings.

(1.2) *Regular presentations.* Assume that $S \cong Q/\mathfrak{b}$ with Q a regular local ring. If the ideal \mathfrak{b} is minimally generated by $\nu_Q \mathfrak{b}$ elements, then

$$(a) \quad \text{cid } S = \nu_Q \mathfrak{b} - \text{grade}_Q S :$$

the equivalent expression $\nu_Q \mathfrak{b} - \text{ht } \mathfrak{b}$ is obtained for $\text{cid } S$ in [12, Satz 1], cf. also [14, (21.1.iii)]; here and below grade denotes the maximal length of a regular sequence in the annihilator of a module, and ht stands for the height of an ideal.

From the Auslander–Buchsbaum Equality for the projective dimension $\text{pd}_R S$, and from the Cohen–Macaulayness of regular rings, one easily sees that

$$(b) \quad \text{cmd } S = \text{pd}_Q S - \text{grade}_Q S .$$

It follows from (b) that S is Cohen–Macaulay if and only if \mathfrak{b} is perfect, that is, when $\text{pd}_R S = \text{grade}_R S$. When this is the case, there is an equality

$$(c) \quad \text{type } S = \text{rank}_\ell \text{Tor}_p^Q(\ell, S) \text{ for } p = \text{pd}_Q S .$$

Indeed, as the ring Q is regular the Tor in (c) is the last non–vanishing homology group of the Koszul complex on a set of generators of \mathfrak{n} . This group is isomorphic to the vector space $((\mathbf{x}) : \mathfrak{n})/(\mathbf{x})$, where \mathbf{x} denotes a maximal S –regular sequence, cf. [2, (1.8)]. Assuming that S is Cohen–Macaulay, \mathbf{x} is a system of parameters, and we see that $\text{rank}_\ell((\mathbf{x}) : \mathfrak{n})/(\mathbf{x})$ does not depend on the choice of the system. On the other hand, this dimension is well known and easily seen to be equal to the number of irreducible \mathfrak{n} –primary components in an irredundant primary decomposition of (\mathbf{x}) , that is, to the type of S .

Note that the preceding argument also establishes the invariance of the number of irreducible ideals in irredundant primary decomposition of parameter ideals in a Cohen–Macaulay local ring.

(1.3) *Flat extensions.* If $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism, then the following hold:

$$(a) [4, (3.6)] \quad \text{cid } S = \text{cid } R + \text{cid } S/\mathfrak{m}S .$$

$$(b) [9, (6.3.2)] \quad \text{cmd } S = \text{cmd } R + \text{cmd } S/\mathfrak{m}S .$$

(c) [11, (1.24)] The ring S is Cohen–Macaulay if and only if R and $S/\mathfrak{m}S$ have this property, and when this is the case there is an equality:

$$\text{type } S = (\text{type } R) \cdot (\text{type } S/\mathfrak{m}S) .$$

Alternative references are provided by [14, (15.1.ii) and (23.3)] for (b); [14, (23.4.Proof)] for (c).

(1.4) *Localizations.* For a prime ideal \mathfrak{q} in S there are equalities:

$$(a) [4, (3.8)] \quad \text{cid } S_{\mathfrak{q}} \leq \text{cid } S .$$

$$(b) [9, (6.11.5)] \quad \text{cmd } S_{\mathfrak{q}} \leq \text{cmd } S .$$

(c) [11, (6.16)] If S is Cohen–Macaulay, then so is $S_{\mathfrak{q}}$ and

$$\text{type } S_{\mathfrak{q}} \leq \text{type } S .$$

When S is complete use Cohen’s Structure Theorem to present it in the form Q/\mathfrak{b} with a regular local ring Q . The inequalities in (a) and (b) then follow because the right–hand

sides of the first two formulas in (1.2) do not increase upon localization; some more care should be exercised in connection with (c), cf. (2.2). For a general ring S , note that by (1.1) its invariants are equal to the corresponding ones of \widehat{S} , then choose by faithful flatness a prime \mathfrak{q}^* in \widehat{S} lying over \mathfrak{q} , and apply (1.3) to the flat local homomorphism $S_{\mathfrak{q}} \rightarrow \widehat{S}_{\mathfrak{q}^*}$.

All local rings of the formal fibers of S are obtained as closed fibers of such homomorphisms by varying \mathfrak{q}^* in $\text{Spec } \widehat{S}$. Thus, when S is a complete intersection (respectively: Gorenstein, Cohen–Macaulay), its formal fibers have the same property, and this yields the more general statement below.

(1.5) *Formal fibers.* If S is a quotient of a complete intersection (respectively: Gorenstein, Cohen–Macaulay) local ring, then its formal fibers are complete intersections (respectively: Gorenstein, Cohen–Macaulay).

In particular, the formal fibers of a complete local ring are complete intersections, and hence also Gorenstein and Cohen–Macaulay. Of course, they are even geometrically regular, cf. e.g. [14, (32.3)], but this will not be needed here.

2. DEVIATION, IMPERFECTION, AND CANONICAL NUMBER

Let (S, \mathfrak{n}, ℓ) be a local noetherian ring. If $S \cong Q/\mathfrak{b}$ for some local ring Q , and the projective dimension $\text{pd}_Q S$ is finite, then motivated by (1.2) we consider the following numbers associated with the Q -module S :

- the *deviation* $\text{dev}_Q S = \text{rank}_{\ell} \text{Tor}_1^Q(\ell, S) - \text{grade}_Q S$; due to the canonical isomorphism $\text{Tor}_1^Q(\ell, S) \cong \ell \otimes_Q \mathfrak{b}$, it can also be defined as $\nu_Q \mathfrak{b} - \text{grade}_Q S$.
- the *imperfection* $\text{imp}_Q S = \text{pd}_Q S - \text{grade}_Q S$;
- the *canonical number* $\text{cnn}_Q S = \text{rank}_{\ell} \text{Tor}_p^Q(\ell, S)$, where $p = \text{pd}_Q S$.

We start with a couple of elementary remarks.

(2.1) *Remark.* $\text{dev}_{\widehat{Q}} \widehat{S} = \text{dev}_Q S$, $\text{imp}_{\widehat{Q}} \widehat{S} = \text{imp}_Q S$, and $\text{cnn}_{\widehat{Q}} \widehat{S} = \text{cnn}_Q S$.

(2.2) *Remark.* If \mathfrak{q} is a prime ideal of S , and $\mathfrak{q}' = \mathfrak{q} \cap Q$ is its inverse image in Q , then

(a)
$$\text{dev}_{Q_{\mathfrak{q}'}} S_{\mathfrak{q}} \leq \text{dev}_Q S.$$

(b)
$$\text{imp}_{Q_{\mathfrak{q}'}} S_{\mathfrak{q}} \leq \text{imp}_Q S.$$

(c) If the Q -module S is perfect, then so is the $Q_{\mathfrak{q}'}$ -module $S_{\mathfrak{q}}$, and

$$\text{cnn}_{Q_{\mathfrak{q}'}} S_{\mathfrak{q}} \leq \text{cnn}_Q S.$$

Indeed, projective dimension does not go up and grade does not go down under localization: this establishes (b), and shows that if S is a perfect module of grade p over Q , then $S_{\mathfrak{q}}$ has the same property over $Q_{\mathfrak{q}'}$. If \mathbf{F} is a minimal free resolution of the Q -module S , then $\mathbf{F}_{\mathfrak{q}'}$ is free resolution of the $Q_{\mathfrak{q}'}$ -module $S_{\mathfrak{q}}$; since $\text{Tor}_n^Q(\ell, S) \cong \ell \otimes_Q F_n$ for $n \in \mathbb{Z}$, there are (well known) inequalities:

$$\text{rank}_{k(\mathfrak{q}')} \text{Tor}_n^{Q_{\mathfrak{q}'}}(k(\mathfrak{q}'), S_{\mathfrak{q}}) \leq \text{rank}_{\ell} \text{Tor}_n^Q(\ell, S)$$

which now take care of (a) and the remaining part of (c)

Next we show that these numbers are invariant under deformation.

(2.3) **Proposition.** *Let $P \xrightarrow{\pi} Q \xrightarrow{\varkappa} S$ be surjective homomorphisms of local rings, such that $\text{Ker } \pi$ is generated by a P -regular sequence.*

When $\text{pd}_Q S$ is finite there are equalities:

- (a) $\text{dev}_P S = \text{dev}_Q S.$
- (b) $\text{imp}_P S = \text{imp}_Q S.$
- (c) $\text{cnn}_P S = \text{cnn}_Q S.$

Proof. It suffices to establish the equalities when $Q = P/(x)$ for a non zero-divisor x in the maximal ideal \mathfrak{r} of P .

Consider the standard change of rings spectral sequence:

$${}^2E_{pq} = \text{Tor}_p^Q(\ell, \text{Tor}_q^P(Q, S)) \Rightarrow \text{Tor}_{p+q}^P(\ell, S).$$

Since $\text{Tor}_q^P(Q, S)$ vanishes for $q \neq 0, 1$, and is isomorphic to S for $q = 0, 1$, it shows that $\text{pd}_P S = p + 1$ when $p = \text{pd}_Q S$, and also provides a ‘‘corner’’ isomorphism $\text{Tor}_{p+1}^P(\ell, S) \cong \text{Tor}_p^Q(\ell, S)$, which settles (c).

The obvious equality $\text{grade}_P S = \text{grade}_Q S + 1$ now yields (b).

To establish (a), we must show that the minimal number of generators of $\text{Ker}(\varkappa\pi)$ is one more than that of $\text{Ker}(\varkappa)$. If this is not the case, then the non zero-divisor x is contained in $\mathfrak{r}\text{Ker}(\varkappa\pi)$. In such a situation [20, Theorem 5] yields isomorphisms:

$$\text{Tor}_n^Q(\ell, S) \cong \bigoplus_{i \geq 0} \text{Tor}_{n-2i}^P(\ell, S) \text{ for } n \in \mathbb{Z},$$

which can only hold when $\text{pd}_Q S = \infty$.

However, this is ruled out by our hypothesis. □

The module invariants considered above enter the proof of the Main Theorem through their links with the ring invariants of fibers of flat homomorphisms. The bridge is provided by Cohen factorizations of local homomorphisms, whose definition we recall next.

A local homomorphism is said to be *weakly regular* if it is flat and its closed fiber is a regular ring. It is proved in [6, (1.1)] that for any local homomorphism $\varphi: R \rightarrow S$ the composition $\hat{\varphi}: R \rightarrow \hat{R} \xrightarrow{\hat{\varphi}} \hat{S}$ factors as:

$$\begin{array}{ccc} & R' & \\ \varphi \nearrow & & \searrow \varphi' \\ R & \xrightarrow{\hat{\varphi}} & \hat{S}, \end{array}$$

with (R', \mathfrak{m}') a complete local ring, $\hat{\varphi}$ a weakly regular homomorphism, and φ' a surjective homomorphism.

Such a decomposition is called a *Cohen factorization* of $\hat{\varphi}$.

The next proposition is a crucial ingredient in the proof of the Main Theorem. In [6, (2.8) and (3.6)] its part (b) is derived from more general results; the proof given below avoids, in particular, an indirect use of the New Intersection Theorem.

(2.4) **Proposition.** *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism.*

For any Cohen factorization $R \rightarrow R' \xrightarrow{\varphi'} \widehat{S}$ of φ the projective dimension of the R' -module \widehat{S} is finite, and the following hold:

$$(a) \quad \text{cid}(S/\mathfrak{m}S) = \text{dev}_{R'} \widehat{S}.$$

$$(b) \quad \text{cmd}(S/\mathfrak{m}S) = \text{imp}_{R'} \widehat{S}.$$

(c) *The ring $S/\mathfrak{m}S$ is Cohen–Macaulay if and only if the R' -module \widehat{S} is perfect, and then*

$$\text{type}(S/\mathfrak{m}S) = \text{cnn}_{R'} \widehat{S}.$$

Proof. By (1.1) we may assume S is complete. We write S instead of \widehat{S} , set $k = R/\mathfrak{m}$, and denote by $\overline{\varphi}'$ the surjective homomorphism $k \otimes_R \varphi'$.

Let \mathbf{F} be an R' -free resolution of S . Using first the flatness of R' over R , and then the flatness of S over R , we get: $H_i(k \otimes_R \mathbf{F}) = \text{Tor}_i^{R'}(k, S) = 0$ for $i > 0$. Thus, the complex $k \otimes_R \mathbf{F}$ is a free resolution of $\overline{S} = S/\mathfrak{m}S$ over $\overline{R}' = R'/\mathfrak{m}R'$. It follows that for any integer n there are isomorphisms:

$$(2.4.1) \quad \text{Tor}_n^{R'}(\ell, S) = H_n(\ell \otimes_{R'} \mathbf{F}) \cong H_n(\ell \otimes_{\overline{R}'} (k \otimes_R \mathbf{F})) = \text{Tor}_n^{\overline{R}'}(\ell, \overline{S}).$$

As a first consequence, we obtain:

$$(2.4.2) \quad \text{pd}_{R'} S = \text{pd}_{\overline{R}'} \overline{S}.$$

Since the ring \overline{R}' is regular, the latter number is finite, say equal to p .

Choose next in $\text{Ker } \overline{\varphi}'$ an \overline{R}' -regular sequence of length $g = \text{grade}_{\overline{R}'} \overline{S}$. As $\text{Ker } \overline{\varphi}' = (\text{Ker } \varphi') \overline{R}'$, it can be lifted to a length g sequence in $\text{Ker } \varphi'$. Since R' is R -flat, this sequence is necessarily R' -regular by [14, (22.5. Corollary)]. Thus, $\text{grade}_{R'} S \geq g = \text{grade}_{\overline{R}'} \overline{S}$. On the other hand, there are relations:

$$\begin{aligned} \text{grade}_{R'} S &\leq \dim R' - \dim S \\ &= (\dim R + \dim \overline{R}') - (\dim R + \dim \overline{S}) \\ &= \dim \overline{R}' - \dim \overline{S} \\ &= \text{grade}_{\overline{R}'} \overline{S}, \end{aligned}$$

where the first equality stems from the flatness of R' and S over R , and the third one from the Cohen–Macaulayness of the regular local ring \overline{R}' . Thus, we have:

$$(2.4.3) \quad \text{grade}_{R'} S = \text{grade}_{\overline{R}'} \overline{S}.$$

By comparing (2.4.3) with (2.4.1) for $n = 1$, one sees that $\text{dev}_{R'} S = \text{dev}_{\overline{R}'} \overline{S}$. Since the ring \overline{R}' is regular, (1.2.a) shows that $\text{dev}_{\overline{R}'} \overline{S} = \text{cid } \overline{S}$, and (a) follows.

Similarly, for (b) compare (2.4.2) and (2.4.3) to obtain $\text{imp}_{R'} S = \text{imp}_{\overline{R}'} \overline{S}$, and then apply (1.2.b) to get $\text{imp}_{\overline{R}'} \overline{S} = \text{cmd } \overline{S}$.

It follows from (b) that the perfection of the R' -module S is equivalent to the Cohen–Macaulayness of the ring \overline{S} , which we now assume. In this case (2.4.1) with $n = p$ yields $\text{cnn}_{R'} S = \text{cnn}_{\overline{R}'} \overline{S}$, and then (1.2.c) shows that the last number equals $\text{type } \overline{S}$, as required for (c).

The proof of the proposition is now complete. \square

A consequence of (a) will be needed at an early stage of the proof of the Main Theorem, so we note it here explicitly:

(2.5) **Corollary.** *If $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of complete local rings, and the closed fiber of φ is a complete intersection, then in any Cohen factorization $R \rightarrow R' \xrightarrow{\varphi'} S$ the kernel of φ' is generated by a regular sequence.* \square

3. PROOF OF THE MAIN THEOREM

For this section we fix the following notation: $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of local rings, \mathfrak{q} is a prime ideal of S , $\mathfrak{p} = \mathfrak{q} \cap R$ is the inverse image of \mathfrak{q} in R , and $\varphi_{\mathfrak{q}}: R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$ is the induced local homomorphism.

First we establish a special case of the Localization Theorem for complete intersections.

Step 1. *If φ is weakly regular and both rings R and S are complete, then the closed fiber of $\varphi_{\mathfrak{q}}$ is a complete intersection.*

Proof. If both R and S are regular, then the closed fiber of $\varphi_{\mathfrak{q}}$ is the quotient of the regular local ring $S_{\mathfrak{q}}$ by the extension of the maximal ideal of the regular local ring $R_{\mathfrak{p}}$. The latter is generated by a regular sequence. Due to the flatness of $\varphi_{\mathfrak{q}}$, so is the former. Thus, the closed fiber of $\varphi_{\mathfrak{q}}$ is a complete intersection.

In the general case, choose by Cohen's Structure Theorem a surjective homomorphism $\pi: P \rightarrow R$ from a regular local ring P . By [6, (1.6)] there is a commutative diagram of local homomorphisms:

$$\begin{array}{ccc} P & \xrightarrow{\zeta} & Q \\ \pi \downarrow & & \downarrow \varkappa \\ R & \xrightarrow{\varphi} & S \end{array}$$

where Q is complete, ζ is weakly regular, \varkappa is surjective, and the induced map $R \otimes_P Q \rightarrow S$ is an isomorphism.

It follows in particular that the ring Q is regular, cf. [14, (23.7.ii)], and that the rings $k(\mathfrak{q} \cap P) \otimes_P Q$ and $k(\mathfrak{q} \cap R) \otimes_R S$ are canonically isomorphic. This isomorphism induces one of the closed fiber of $\zeta_{\mathfrak{q} \cap Q}$ with that of $\varphi_{\mathfrak{q}}$, so that the assertion of the lemma follows from the special case already settled above. \square

Next we prove the theorem in the complete case, when the invariants of the formal fibers are all trivial by (1.5).

Step 2. *If R and S are complete local rings, then there are inequalities:*

- (a) $\text{cid } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \leq \text{cid } S/\mathfrak{m}S.$
- (b) $\text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \leq \text{cmd } S/\mathfrak{m}S.$
- (c) *If furthermore $S/\mathfrak{m}S$ is Cohen–Macaulay, then so is $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ and*
 $\text{type } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \leq \text{type } S/\mathfrak{m}S.$

Proof. The arguments will be given for Cohen–Macaulay defects. The other two assertions are obtained by manual or electronic changes in notation, accompanied by the corresponding switches to parts (a) or (c) of the quoted results.

Take a Cohen factorization $\varphi: R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, set $\mathfrak{p}' = \mathfrak{q} \cap R$, and consider the induced factorization

$$\varphi_{\mathfrak{q}}: R_{\mathfrak{p}} \xrightarrow{\dot{\varphi}_{\mathfrak{p}'}} R'_{\mathfrak{p}'} \xrightarrow{\varphi'_{\mathfrak{q}}} S_{\mathfrak{q}}.$$

Writing \tilde{R} , \tilde{R}' , and \tilde{S} for the completions of the local rings $R_{\mathfrak{p}}$, $R'_{\mathfrak{p}'}$, and $S_{\mathfrak{q}}$ in their respective maximal-ideal-adic topologies, we obtain flat local homomorphisms:

$$\tilde{\varphi}: \tilde{R} \xrightarrow{\psi} \tilde{R}' \xrightarrow{\vartheta} \tilde{S}.$$

Denoting by $\tilde{\mathfrak{p}}$ the maximal ideal of \tilde{R} and remarking that the closed fiber of $\tilde{\varphi}$ is the completion of that of $\varphi_{\mathfrak{q}}$, we record the equality:

$$\text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} = \text{cmd } \tilde{S}/\tilde{\mathfrak{p}}\tilde{S}.$$

Next we take a Cohen factorization

$$\tilde{R} \xrightarrow{\dot{\psi}} \tilde{R}'' \xrightarrow{\psi'} \tilde{R}'$$

of ψ , and note that $(\vartheta\psi')\dot{\psi}$ is one of $\tilde{\varphi}$. Thus, (2.4.b) yields:

$$\text{cmd } \tilde{S}/\tilde{\mathfrak{p}}\tilde{S} = \text{imp}_{\tilde{R}''} \tilde{S}.$$

The closed fiber $\tilde{R}'/\tilde{\mathfrak{p}}\tilde{R}'$ of ψ is the completion of that of $\dot{\varphi}_{\mathfrak{p}'}$. By Step 1 the latter is a complete intersection, hence so is the former. By (2.5) the kernel of ψ' is generated by an \tilde{R}'' -regular sequence. Applications of (2.3.b), (2.1), and (2.2.b) yield:

$$\text{imp}_{\tilde{R}''} \tilde{S} = \text{imp}_{\tilde{R}'} \tilde{S} = \text{imp}_{R'_{\mathfrak{p}'}} S_{\mathfrak{q}} \leq \text{imp}_{R'} S.$$

Referring again to (2.4.b) we get:

$$\text{imp}_{R'} S = \text{cmd } S/\mathfrak{m}S.$$

To obtain the assertion of Step 2(b) it only remains to concatenate the (in)equalities above. \square

Finally, we deal with the general case.

Step 3. For any φ there are inequalities:

- (a) $\text{cid } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cid}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{cid } S/\mathfrak{m}S + \text{cid}(k(\mathfrak{p}) \otimes_R \widehat{R}) .$
 (b) $\text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{cmd } S/\mathfrak{m}S + \text{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}) .$

(c) If $S/\mathfrak{m}S$ and the formal fibers of R at \mathfrak{p} are Cohen–Macaulay, then $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ and the formal fibers of S at \mathfrak{q} are Cohen–Macaulay, and there is an inequality:

$$\text{type}(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) \cdot \text{type}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \text{type}(S/\mathfrak{m}S) \cdot \text{type}(k(\mathfrak{p}) \otimes_R \widehat{R}) .$$

Proof. Here again, we write down the argument for (b) only, and refer to the beginning of the preceding proof for details on the treatment of the other two assertions.

Choose by faithful flatness a prime ideal $\mathfrak{q}^* \in \text{Spec } \widehat{S}$ which lies over $\mathfrak{q} \in \text{Spec } S$, and set $\mathfrak{p}^* = \mathfrak{q}^* \cap \widehat{R}$. By the additivity (1.3.b) of Cohen–Macaulay defects on local flat extensions we now have the following simple computation:

$$\begin{aligned} \text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cmd } \widehat{S}_{\mathfrak{q}^*}/\mathfrak{q}\widehat{S}_{\mathfrak{q}^*} &= (\text{cmd } S_{\mathfrak{q}} - \text{cmd } R_{\mathfrak{p}}) + (\text{cmd } \widehat{S}_{\mathfrak{q}^*} - \text{cmd } S_{\mathfrak{q}}) \\ &= \text{cmd } \widehat{S}_{\mathfrak{q}^*} - \text{cmd } R_{\mathfrak{p}} \\ &= (\text{cmd } \widehat{R}_{\mathfrak{p}^*} - \text{cmd } R_{\mathfrak{p}}) + (\text{cmd } \widehat{S}_{\mathfrak{q}^*} - \text{cmd } \widehat{R}_{\mathfrak{p}^*}) \\ &= \text{cmd } \widehat{R}_{\mathfrak{p}^*}/\mathfrak{p}\widehat{R}_{\mathfrak{p}^*} + \text{cmd } \widehat{S}_{\mathfrak{q}^*}/\mathfrak{p}\widehat{S}_{\mathfrak{q}^*} . \end{aligned}$$

Because $\widehat{R}_{\mathfrak{p}^*}/\mathfrak{p}\widehat{R}_{\mathfrak{p}^*}$ is a localization of $k(\mathfrak{p}) \otimes_R \widehat{R}$, we get from (1.4.b) an inequality:

$$\text{cmd } \widehat{R}_{\mathfrak{p}^*}/\mathfrak{p}\widehat{R}_{\mathfrak{p}^*} \leq \text{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}) .$$

On the other hand, the result of Step 2 and (1.1) show that:

$$\text{cmd } \widehat{S}_{\mathfrak{q}^*}/\mathfrak{p}\widehat{S}_{\mathfrak{q}^*} \leq \text{cmd } \widehat{S}/\widehat{\mathfrak{m}}\widehat{S} = \text{cmd } S/\mathfrak{m}S .$$

Using these formulas to replace the corresponding terms of the last sum above, we obtain:

$$\text{cmd } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} + \text{cmd } \widehat{S}_{\mathfrak{q}^*}/\mathfrak{q}\widehat{S}_{\mathfrak{q}^*} \leq \text{cmd } S/\mathfrak{m}S + \text{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}) .$$

By taking the supremum over the prime ideals $\mathfrak{q}^* \in \text{Spec } \widehat{S}$ which lie over \mathfrak{q} , we see that part (b) of the Main Theorem follows from the last inequality. \square

4. LOCALIZATION THEOREMS

We first note a very special case of the Main Theorem.

(4.1) **Theorem.** *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of local rings. Assume that the formal fibers of R and the closed fiber $S/\mathfrak{m}S$ of φ have one of the following properties:*

- (CI) *complete intersection.*
- (G) *Gorenstein.*
- (CM) *Cohen–Macaulay.*

All fibers of φ and formal fibers of S then have the corresponding property. □

(4.2) *Remark.* Assume that the formal fibers of R have property (CI) (respectively: (G), (CM)) above. The Main Theorem shows that if for some integer n the closed fiber $S/\mathfrak{m}S$ has complete intersection defect at most n (respectively: is Cohen–Macaulay of type at most n , has Cohen–Macaulay defect at most n), then the latter condition holds for all fibers of φ and all formal fibers of S .

Next we refine the Localization Theorem (4.1) by involving Serre’s condition (S_n) and its variants, like Ischebeck’s condition (G_n) . Let A be a noetherian ring. For a fixed integer n , consider the properties:

- (CI_n) if $\mathfrak{p} \in \text{Spec } A$ has $\text{depth } A_{\mathfrak{p}} < n$, then $A_{\mathfrak{p}}$ is a complete intersection.
- (G_n) if $\mathfrak{p} \in \text{Spec } A$ has $\text{depth } A_{\mathfrak{p}} < n$, then $A_{\mathfrak{p}}$ is Gorenstein.
- (S_n) if $\mathfrak{p} \in \text{Spec } A$ has $\text{depth } A_{\mathfrak{p}} < n$, then $A_{\mathfrak{p}}$ is Cohen–Macaulay.

Clearly, A has the absolute property \mathcal{P} = “complete intersection” (respectively: “Gorenstein,” “Cohen–Macaulay”) precisely when it has the partial property \mathcal{P}_n = “ (CI_n) ” (respectively: “ (G_n) ,” “ (S_n) ”) for all $n \in \mathbb{Z}$. Rings satisfying (G_n) are known as n -Gorenstein [18]; by restriction of language, we say A is n -complete intersection if it has (CI_n) .

We need information on the behavior of the partial properties \mathcal{P}_n under flat homomorphisms. It is derived from the following facts about the corresponding absolute properties \mathcal{P} , which are limit cases of the equalities (1.3).

(4.3) If $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of local rings, then:

- (1) If R and $S/\mathfrak{m}S$ have \mathcal{P} then S has \mathcal{P} .
- (2) If S has \mathcal{P} then so does R .

When \mathcal{P}_n = “ (S_n) ” the argument for the next lemma is indicated in [9, (7.3.4), (7.3.8)]; it is noted in [17, Proposition 3] that the same pattern applies to \mathcal{P}_n = “ (G_n) .”

(4.4) **Lemma.** *Let $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ be flat homomorphisms of noetherian rings, and let \mathcal{P}_n denote one of the conditions (CI_n) , (G_n) , or (S_n) . The following then hold:*

- (1) *If the fibers of α and β have \mathcal{P}_n then so do the fibers of $\beta\alpha$.*
- (2) *If β is faithfully flat and the fibers of $\beta\alpha$ have \mathcal{P}_n then so do those of α .*

Proof. If k is the quotient field of a localization of A at a prime ideal, the induced homomorphism $k \otimes_A B \rightarrow k \otimes_A C$ is flat (respectively, faithfully flat) along with β , and its fibers are among those of β . Thus, it suffices to show the following, proved in [9, (6.4.1)] for \mathcal{P}_n = “ (S_n) ” by an argument which applies equally well to the other cases, cf. [18, Proposition 1] or [17, Proposition 1] for \mathcal{P}_n = “ (G_n) .”

- (1') If B and the fibers of β satisfy \mathcal{P}_n then C satisfies \mathcal{P}_n .
 (2') If β is faithfully flat and C satisfies \mathcal{P}_n then so does B .

For $\mathfrak{q} \in \text{Spec } C$ with $\text{depth } C_{\mathfrak{q}} < n$, set $\mathfrak{p} = \mathfrak{q} \cap B$ and consider the induced flat local homomorphism $B_{\mathfrak{p}} \rightarrow C_{\mathfrak{q}}$. The additivity of depth on flat extensions,

$$\text{depth } B_{\mathfrak{p}} + \text{depth } C_{\mathfrak{q}}/(\mathfrak{p})C_{\mathfrak{q}} = \text{depth } C_{\mathfrak{q}},$$

cf. e.g. [14, (23.3)], implies that the depth $B_{\mathfrak{p}}$ and $C_{\mathfrak{q}}/(\mathfrak{p})C_{\mathfrak{q}}$ is smaller than n , hence both rings satisfy \mathcal{P} . By (4.3.1) then so does $C_{\mathfrak{q}}$, and this proves (1').

To prove (2'), start from a given $\mathfrak{p} \in \text{Spec } R$ with $\text{depth } B_{\mathfrak{p}} < n$ and choose $\mathfrak{q} \in \text{Spec } C$ minimal over \mathfrak{p} . Using again the additivity of depth, one sees that $\text{depth } C_{\mathfrak{q}} = \text{depth } B_{\mathfrak{p}} < n$, hence $C_{\mathfrak{q}}$ has \mathcal{P} . By (4.3.2) so does $B_{\mathfrak{p}}$. \square

Next we derive the refined form of the Localization Theorem from its primitive form (4.1) by the argument of [15, (2.4.Proof)], repeated in [7, (3.3.Proof)] to obtain a partial Localization Theorem for $\mathcal{P} = \text{"(S}_n\text{"}$ from one for $\mathcal{P} = \text{"Cohen-Macaulay."}$ We reproduce it here for completeness.

(4.5) **Theorem.** *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of local rings. If for some integer n one of the conditions (CI_n) , (G_n) , or (S_n) holds for the formal fibers of R and the closed fiber of φ , then it holds for all fibers of φ .*

Proof. Consider first the case when R is complete. Since its formal fibers then have \mathcal{P} by (1.5), if the Localization Theorem fails there is a prime $\mathfrak{p} \subsetneq \mathfrak{m}$ such that $k(\mathfrak{p}) \otimes_R S$ does not satisfy \mathcal{P}_n . By Noetherian induction choose \mathfrak{p} maximal among such primes. After replacing φ by the induced flat local homomorphism $R/\mathfrak{p} \rightarrow S/\mathfrak{p}S$, we change notation, and assume that R is a domain and all the fibers of φ but the generic one enjoy \mathcal{P}_n .

Thus, there is a prime \mathfrak{q} in S lying over (0) in R , with $\text{depth } S_{\mathfrak{q}} < n$ and for which the ring $(R_{(0)} \otimes_R S)_{\mathfrak{q}}$ does not have \mathcal{P} . Choose a non-zero element $x \in \mathfrak{m}$, and then a prime \mathfrak{q}' in S , minimal over $(\mathfrak{q} + Sx)$. Set $\mathfrak{p}' = \mathfrak{q}' \cap R$, and consider the induced flat local homomorphism $\varphi_{\mathfrak{q}'}: R_{\mathfrak{p}'} \rightarrow S_{\mathfrak{q}'}$. Note that $\text{depth } R_{\mathfrak{p}'} \geq 1$ as $x \in \mathfrak{p}'$, and hence:

$$\text{depth } S_{\mathfrak{q}'}/\mathfrak{p}'S_{\mathfrak{q}'} = \text{depth } S_{\mathfrak{q}'} - \text{depth } R_{\mathfrak{p}'} \leq \text{depth } S_{\mathfrak{q}} + \text{ht}(\mathfrak{q}'/\mathfrak{q}) - \text{depth } R_{\mathfrak{p}'} < n.$$

Because $S_{\mathfrak{q}'}/\mathfrak{p}'S_{\mathfrak{q}'}$ is a localization of the non-generic fiber $k(\mathfrak{p}') \otimes_R S$ of φ , the depth estimate above implies that it satisfies \mathcal{P} . Applying (4.1) we see that all the fibers of $\varphi_{\mathfrak{q}'}$ then have \mathcal{P} . This contradicts the assumption on its generic fiber, and thus finishes the argument for complete R .

For arbitrary R , consider the composition $R \xrightarrow{\rho} \widehat{R} \xrightarrow{\varphi^*} S^*$, where S^* denotes the $(\mathfrak{m}S)$ -adic completion of S . The closed fiber of φ^* is canonically isomorphic with that of φ , hence all the fibers of φ^* satisfy \mathcal{P}_n by the case already settled above. The fibers of ρ satisfy \mathcal{P}_n by assumption. It follows from (4.4.1) that \mathcal{P}_n holds for the fibers of $\varphi^* \rho = \sigma \varphi$, where $\sigma: S \rightarrow S^*$ is the completion map. As σ is faithfully flat, (4.4.2) implies that the fibers of φ have the same property, as was to be proved. \square

5. CONCLUDING REMARKS

Grothendieck's original approach to the Localization Problem begins with a careful analysis in [9, (7.3)] of the behavior under base change of the properties \mathcal{P} under consideration. He then proves Localization Theorems assuming the induced residue field extension $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is finite, [*idem*, (7.5.2)], or assuming the existence of resolution of singularities, [*idem*, (7.9.8)], cf. also [13, (2.1)].

The properties considered in (4.1), (4.2), and (4.5) satisfy Grothendieck's conditions, hence these results are known in the residually finite and characteristic zero cases.

The first unrestricted positive answer to the Localization Problem has been obtained by André [1] for "geometrically regular," using the André–Quillen homology theory of commutative algebras. Nishimura [15] has applied this result to prove Localization Theorems for \mathcal{P} = "geometrically reduced" and \mathcal{P} = "geometrically normal," and his method has been extended by Brezuleanu and Ionescu [7] to higher versions of normality, like "geometrically $(\mathbf{R}_n) + (\mathbf{S}_{n+1})$."

The Localization Problem for \mathcal{P} = "complete intersection" is settled by Tabâ [19], sharpening the method of [13, (4.6)]. Both papers rely on André–Quillen homology and the fact that the analog of (4.4) holds for the complete intersection property, cf. [3]. The proof of (4.1.CI) does not use André–Quillen homology.

The Localization Theorem for \mathcal{P} = "Gorenstein" is reduced by Marot [13, (3.2)] to the complete case, established by Hall and Sharp [10, (3.3)] with the help of dualizing complexes. The proof of (4.1.G) does not depend on that theory.

For \mathcal{P} = "Cohen–Macaulay" and its generalizations \mathcal{P} = " (\mathbf{S}_n) " and \mathcal{P} = "of Cohen–Macaulay defect at most n ," the only previous advance beyond the cases originally settled by Grothendieck is in small dimensions: When the dimension of R is at most 4 the original argument can be adapted to use "macaulayfications" in place of resolutions of singularities, cf. [7, (3.1), (3.3)]. The results of (4.1.CM), (4.2), and (4.5. \mathbf{S}_n) provide the first unrestricted Localization Theorems for these properties.

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