

GAPS IN HOCHSCHILD COHOMOLOGY IMPLY SMOOTHNESS FOR COMMUTATIVE ALGEBRAS

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ABSTRACT. The paper concerns Hochschild cohomology of a commutative algebra S , which is essentially of finite type over a commutative noetherian ring K and projective as a K -module. For a finite S -module M it is proved that vanishing of $\mathrm{HH}^n(S|K; M)$ in sufficiently long intervals imply the smoothness of $S_{\mathfrak{q}}$ over K for all prime ideals \mathfrak{q} in the support of M . In particular, S is smooth if $\mathrm{HH}^n(S|K; S) = 0$ for $(\dim S + 2)$ consecutive $n \geq 0$.

INTRODUCTION

Let K be a commutative noetherian ring, S a commutative K -algebra, and M an S -module. We let $\mathrm{HH}_*(S|K; M)$ and $\mathrm{HH}^*(S|K; M)$ denote, respectively, the Hochschild homology and the Hochschild cohomology of the K -algebra S with coefficients in M . For each $n \in \mathbb{Z}$ there are canonical homomorphisms

$$\begin{aligned} \lambda_n^M : (\wedge_S^n \Omega_{S|K}) \otimes_S M &\longrightarrow \mathrm{HH}_n(S|K; M) \\ \lambda_M^n : \mathrm{HH}^n(S|K; M) &\longrightarrow \mathrm{Hom}_S(\wedge^n \Omega_{S|K}, M) \end{aligned}$$

of S -modules, where $\Omega_{S|K}$ is the S -module of K -linear Kähler differentials of S . Other concepts appearing the next result are defined following its statement.

Main Theorem. *Let K be a commutative noetherian ring and S a commutative K -algebra essentially of finite type, flat as a K -module. For a prime ideal \mathfrak{q} in S and a finite S -module M with $M_{\mathfrak{q}} \neq 0$ the following conditions are equivalent:*

- (i) *The K -algebra $S_{\mathfrak{q}}$ is smooth.*
- (ii_{*}) *Each map $(\lambda_n^M)_{\mathfrak{q}}$ is bijective and the $S_{\mathfrak{q}}$ -module $\Omega_{S_{\mathfrak{q}}|K}$ is projective.*
- (iii_{*}) *There exist non-negative integers t, u of different parity satisfying*

$$\mathrm{HH}_t(S|K; M)_{\mathfrak{q}} = 0 = \mathrm{HH}_u(S|K; M)_{\mathfrak{q}}$$

When the K -module S is projective they are also equivalent to:

- (ii^{*}) *Each map $(\lambda_M^n)_{\mathfrak{q}}$ is bijective.*
- (iii^{*}) *There exist non-negative integers t, u of different parity satisfying*

$$\mathrm{HH}^{t+i}(S|K; M)_{\mathfrak{q}} = 0 = \mathrm{HH}^{u+i}(S|K; M)_{\mathfrak{q}} \quad \text{for } 0 \leq i \leq \dim_{S_{\mathfrak{q}}} M_{\mathfrak{q}}$$

Recall that one says that S is *essentially of finite type* if it is a localization of a finitely generated K -algebra. A flat K -algebra S essentially of finite type is *smooth* if the structure map $K \rightarrow S$ has geometrically regular fibers. Equivalently, for every homomorphism of rings $K \rightarrow \ell$, where ℓ is field, the ring $S \otimes_K \ell$ has finite

Date: July 31, 2005.

2000 Mathematics Subject Classification. Primary 13D03, 14B25. Secondary 14M10, 16E40.
Research partly supported by NSF grants DMS 0201904 (L.L.A) and DMS 0442242 (S.I.).

global dimension. We say that an S -module M is *finite* if it is finitely generated, and let $\dim_S M$ denote the Krull dimension of M .

The theorem incorporates several known results, discussed below. There are two new aspects to our characterizations of smoothness: the use of *cohomology* (with a couple of exceptions, earlier results used vanishing of homology) and the introduction of *coefficients* (all earlier results dealt with the case $M = S$). A special case of the theorem relates to a question of Happel [16, (1.4)]:

For a (not necessarily commutative) algebra A over a field K , with $\text{rank}_K A$ finite, does $\text{HH}^n(A|K; A) = 0$ for $n \gg 0$ imply finite global dimension?

The next corollary provides a strong affirmative answer in the commutative case. This is in sharp contrast to the general situation, where the answer is negative: see the companion paper [11] by Buchweitz, Green, Madsen, and Solberg.

Corollary. *Let K be a field let S be a commutative K -algebra, finite dimensional as a K -vector space. If $\text{HH}^n(S|K; S) = 0$ for two non-negative values of n of different parity, then S is a product of separable field extensions of K .*

Proof. The hypothesis $\text{rank}_K S < \infty$ implies that $\dim S$ is 0, and that S is smooth precisely when it is a product of finite separable field extensions of K . \square

We place our result in the context of earlier work relating vanishing of Hochschild (co)homology and smoothness. As always, $\text{Spec } S$ denotes the set of prime ideal of S ; its subset $\text{Supp}_S M = \{\mathfrak{q} \in \text{Spec } S \mid M_{\mathfrak{q}} \neq 0\}$ is the *support* of M .

Antecedents. Let S be a K -algebra S essentially of finite type, flat as a K -module. When citing results, a roman numeral in *italic font* indicates the variant of the correspondingly numbered condition in the Main Theorem, where the hypothesis is assumed to hold for $M = S$ and for all $\mathfrak{q} \in \text{Spec } S$.

The HKR Theorem. Hochschild, Kostant, and Rosenberg [18] (when K is a perfect field) and André [1] (in general) proved (i) \implies (ii $_*$) & (ii *). As S is essentially of finite type, the S -module $\Omega_{S|K}$ is finite, so $\wedge_S^n \Omega_{S|K} = 0$ holds for all $n \gg 0$, hence one always has (ii $_*$) \implies (iii $_*$) and (ii *) \implies (iii *).

Homological converses to the HKR Theorem. André [1] proved (ii $_*$) \implies (i).

(iii $_*$) \implies (i) was proved by Avramov and Vigué-Poirrier [6] when K is a field; by Campillo, Guccione, Guccione, Redondo, Solotar, and Villamayor [7] when, in addition, $\text{char}(K) = 0$; by Rodicio [22] in general.

Cohomological converses to the HKR Theorem. Assume S is projective over K .

For a Gorenstein ring S Blanco and Majadas [8] proved that $\text{HH}^n(S|K; S) = 0$ for $(\dim S + 2)$ consecutive values of $n \geq 0$ implies S is smooth over K ; this is subsumed in the implication (iii *) \implies (i) of the Main Theorem. In joint work with Rodicio [9] they showed that if S is locally complete intersection over K , then $\text{HH}^{2n}(S|K; S) = 0$ or $\text{Ker}(\lambda_S^{2n}) = 0$ for a single $n \geq 0$ implies S is smooth.

Generalizations. The Main Theorem is a special case of a much more general result, Theorem (4.2), concerning gaps in $\text{Tor}_*^R(S, M)$ and $\text{Ext}_R^*(S, N)$ when R is a noetherian ring, S is an algebra retract of R , and M is a *complex* of S -modules. For $\text{Tor}_*^R(S, S)$ that result is due to Rodicio [22]. However, to prove (iii *) \implies (i) in the Main Theorem, even for $M = S$, we do need to use complexes.

Conventions. In the rest of this article all rings are assumed to be commutative. A *local ring* is a noetherian ring that has a unique maximal ideal. A *local homomorphism* is a homomorphism of rings, whose source and target are local and which maps maximal ideal into maximal ideal.

1. CLOSED HOMOMORPHISMS

In this section $\varphi: (R, \mathfrak{m}, k) \rightarrow S$ is a surjective local homomorphism.

We recall a general construction due to Tate [24]. More details about Tate resolutions and acyclic closures can be found in the original paper, in the book of Gulliksen and Levin [14, Chapter I], or in the survey [2, Chapter 6].

1.1. Tate resolutions. For each positive integer n let X_n denote a free graded R -module concentrated in degree n ; furthermore, $R\langle X_n \rangle$ denotes the exterior algebra on X_n if n is odd, and the divided powers algebra on X_n if n is even; in the latter case, $x^{(i)}$ denotes the i th divided power of $x \in X_n$.

A *Tate resolution* of φ is a DG (= differential graded) algebra G having a system of divided powers compatible with the action of the differential and a filtration $\{G^{(p)}\}_{p \geq 0}$ by DG subalgebras with divided powers, such that

- (0) $G^{(0)} = R$ and $G^{(p-1)} \subseteq G^{(p)}$, for $p \geq 1$.
- (1) $G^{(p)} = G^{(p-1)} \otimes_R R\langle X_p \rangle$ as graded R -modules, for $p \geq 1$.
- (2) $\partial(x^{(i)}) = \partial(x)x^{(i-1)}$ for all $i \geq 1$ when $|x|$ is even and positive.
- (3) $H_0(G^{(p)}) = S$ for $p \geq 1$.
- (4) $H_i(G^{(p)}) = 0$ for $1 \leq i < p$.
- (5) $G = \bigcup_{p \geq 0} G^{(p)}$.

Forgetting the multiplicative structures, G is a free resolution of R over S . One always exists: form DG algebras satisfying conditions (0) through (4) by induction on p , then use (5) to define G . Control may be exercised at each step of the process.

As starting point, one may choose any surjective R -linear map

$$\delta_1: X_1 \longrightarrow \text{Ker}(\varphi)$$

and define the differential on $G^{(1)}$ so that its restriction to X_1 is the composition of δ_1 with the inclusion $\text{Ker}(\varphi) \subseteq R = G_0^{(0)}$. If e_1 is a basis for X_1 , then $\delta_1(e_1)$ generates the ideal $\text{Ker}(\varphi)$ and $G^{(1)}$ is the Koszul complex on $\delta(e_1)$.

For each $p \geq 2$ one may choose any surjective R -linear map

$$\delta_p: X_p \rightarrow H_{p-1}(G^{(p-1)})$$

lift it to a homomorphism $\tilde{\delta}_p: X_p \rightarrow Z_{p-1}(G^{(p-1)})$, and define a differential on $G^{(p)}$, which on X_p is the composition of $\tilde{\delta}_p$ with $Z_{p-1}(G^{(p-1)}) \subseteq G_{p-1}^{(p-1)} = G_{p-1}^{(p)}$.

1.2. Acyclic closures. An *acyclic closure* of φ is a Tate resolution obtained by choosing for each $p \geq 1$ the map δ_p in (1.1) to be a projective cover.

Let G be an acyclic closure of φ and let G' be a Tate resolution of φ . There exists then a morphism $\gamma: G \rightarrow G'$ of DG R -algebras with divided powers, and for any such morphism the homomorphism of R -modules $\gamma_n: G_n \rightarrow G'_n$ is a split injection. If G' is also an acyclic closure of φ , then γ is an isomorphism, and it induces an isomorphism $G^{(p)} \rightarrow G'^{(p)}$ for each $p \geq 0$.

In particular, the p th stage $G^{(p)}$ of an acyclic closure G of φ is independent, up to isomorphism, of the choice of G .

The next remark is immediate from the construction of acyclic closures.

1.2.1. $G^{(1)}$ is the Koszul complex E on a minimal generating set for $\text{Ker}(\varphi)$.

We introduce two numerical invariants of φ , for use throughout the paper. Letting $\nu_S(N)$ denote the minimal number of generators an S -module N , we set

$$\varepsilon_2(\varphi) = \nu_S(\text{Ker}(\varphi)) \quad \text{and} \quad \varepsilon_3(\varphi) = \nu_S(\text{H}_1(E))$$

These are part of the *deviations* of φ ; see [5, (2.5)]. The first assertion below is clear; the second one is a standard characterization of regular sequences.

1.2.2. $\varepsilon_2(\varphi) = 0$ if and only if $\varphi = \text{id}^R$.

1.2.3. $\varepsilon_3(\varphi) = 0$ if and only if φ is generated by a regular sequence.

A complex F of finite free R -modules is said to be *minimal* if $\partial(F) \subseteq \mathfrak{m}F$. For each integer $p \geq 1$, the construction of the p th stage $F_{\leq p}$ of a minimal resolution of S adds to $F_{\leq p-1}$ a single new free module in degree p .

In contrast, the construction of the p th stage $G^{(p)}$ of an acyclic closure of φ adds to $G^{(p-1)}$ shifts of *every* free module present in it: finitely many shifts appear when p is odd, and infinitely many when p is even. Thus, when the resolution of S over R provided by an acyclic closure is minimal, one has a certain control of the growth of that resolution.

This explains our interest in the class of maps described below.

1.3. Closed homomorphisms. We say that the homomorphism φ is *closed* if some acyclic closure G of φ is a minimal resolution of S over R .

A celebrated result of Gulliksen [13] and Schoeller [23] can be read as follows:

1.3.1. The canonical surjection $R \rightarrow k$ is closed for every R .

To state an extension, we recall that the homomorphism φ is *large* if the map

$$\text{Tor}_n^\varphi(k, k): \text{Tor}_n^R(k, k) \rightarrow \text{Tor}_n^S(k, k)$$

is surjective for each n . The notion was introduced by Levin [20]. The following theorem of Avramov and Rahbar-Rochandel, see [20, (2.5)], provides a significant supply of closed homomorphisms.

1.3.2. Every large homomorphism is closed.

The last result will be applied through the following observation:

1.3.3. If there is a homomorphism of rings $\psi: S \rightarrow R$ with $\varphi \circ \psi = \text{id}^S$, then

$$\text{Tor}_n^\varphi(k, k) \circ \text{Tor}_n^\psi(k, k) = \text{Tor}_n^{\varphi \circ \psi}(k, k) = \text{Tor}_n^{\text{id}^S}(k, k) = \text{id}^{\text{Tor}_n^S(k, k)}$$

by functoriality; thus, $\text{Tor}_n^\varphi(k, k)$ is surjective, hence φ is large, and so closed.

In this paper we are mostly interested in obtaining lower bounds on the sizes of the S -modules $\text{Tor}_n^R(S, M)$ and $\text{Ext}_R^n(S, M)$. For that purpose we use properties of φ that are weaker than closure.

1.4. Partly closed homomorphisms. Let $G^{(p)}$ be as in (1.2) for some $p \geq 1$ and F be a minimal free resolution of the R -module S . As $\text{H}_0(G^{(p)}) = S$ and each $G_n^{(p)}$ is R -free, the augmentation $G^{(p)} \rightarrow S$ lifts to a *comparison morphism*

$$\gamma^{(p)}: G^{(p)} \rightarrow F$$

We say that φ is *p -closed* if $\gamma_n^{(p)}$ has an R -linear left inverse for each $n \in \mathbb{Z}$.

A homomorphism γ of free R -modules of finite rank has a left inverse if and only if the map $k \otimes_R \gamma$ is injective. This yields an alternative description:

1.4.1. The homomorphism φ is p -closed if and only if $G^{(p)}$ is minimal and the induced map $H(k \otimes_R \gamma^{(p)}): k \otimes_R G^{(p)} \rightarrow \text{Tor}^R(k, S)$ is injective.

1.4.2. If the homomorphism φ is p -closed, G' an acyclic closure of φ , F' is a resolution of S by finite free R -modules, and $\gamma'^{(p)}: G'^{(p)} \rightarrow F'$ is a comparison morphism, then $\gamma'_n^{(p)}$ has a left inverse for each $n \in \mathbb{Z}$.

Indeed, (1.2) yields an isomorphism $\alpha: G^{(p)} \rightarrow G'^{(p)}$ of DG algebras over R , so $G'^{(p)}$ is minimal by (1.4.1). For any comparison morphism $\beta: F' \rightarrow F$, the morphisms $\gamma^{(p)}$ and $\beta\gamma'^{(p)}\alpha$ are homotopic. Thus, $H(k \otimes_R \gamma^{(p)})$ factors as

$$k \otimes_R G^{(p)} \xrightarrow{k \otimes_R \alpha} k \otimes_R G'^{(p)} \xrightarrow{H(k \otimes_R \gamma'^{(p)})} H(k \otimes_R F') \xrightarrow{H(k \otimes_R \beta)} k \otimes_R F$$

It follows that $H(k \otimes_R \gamma^{(p)})$ is injective, see (1.4.1), hence $\gamma_n^{(p)}$ is split injective.

1.4.3. Let R' be a local ring, let $\rho: R \rightarrow R'$ be a faithfully flat local homomorphism, set $S' = R' \otimes_R S$, and let $\rho': R' \rightarrow S'$ denote the induced homomorphism. The map φ is p -closed if and only if so is φ' .

Indeed, $R' \otimes_R G$ is an acyclic closure of φ' if G is one of φ ; see [14, (1.9.8)].

Next we place the properties discussed above in a familiar context, focusing on the case $p \leq 2$ because these are the classes of maps important for this paper.

1.5. Comparisons. The homomorphism φ is said to be *complete intersection* (or *c.i.*, for short) if $\text{Ker}(\varphi)$ is generated by a regular sequence. Clearly, one has

$$\text{c.i.} \implies \text{closed} \implies \text{2-closed} \implies \text{1-closed}$$

1.5.1. The first implication is obviously strict; see for instance (1.3.1).

1.5.2. The canonical map from $R = k[[x, y]]/(x^2, xy)$ to $S = R/(y^2)$ is 1-closed, but not 2-closed: apply (1.4.2) to the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{x} & R & \xrightarrow{y^2} & R & \xrightarrow{x} & R & \xrightarrow{y^2} & R & \longrightarrow & 0 \\ & & \downarrow \begin{bmatrix} 0 \\ y \end{bmatrix} & & \parallel & & \parallel & & \parallel & & \\ \cdots & \longrightarrow & R^2 & \xrightarrow{[x \ y]} & R & \xrightarrow{x} & R & \xrightarrow{y^2} & R & \longrightarrow & 0 \end{array}$$

whose top row is the beginning of the second stage of an acyclic closure of φ and whose bottom row is the beginning of a minimal resolution of S over R .

1.5.3. We do not whether every 2-closed homomorphisms is actually closed.

Except for the name, 1-closed homomorphisms have appeared in literature.

1.6. One-closed homomorphisms. As defined, p -closure requires $\gamma_n^{(p)}$ to be split injective for each n . However, 1-closure can be detected in a single degree.

Lemma 1.6.1. *Let E be the Koszul complex on a minimal generating set for the ideal $\text{Ker}(\varphi)$ and let $\gamma: E \rightarrow F$ be a comparison morphism to a minimal free resolution of S over R . The following conditions are equivalent:*

- (1) *The homomorphism φ is 1-closed.*

(2) For $c = \varepsilon_2(\varphi)$ the map $(k \otimes_R \gamma_c): (k \otimes_R E_c) \rightarrow \mathrm{Tor}_c^R(k, S)$ is injective.

Proof. Observation (1.2.1) and property (1.4.1) show that (1) implies (2).

For the converse, note that the isomorphism $k \otimes_R E \cong \wedge k^c$ of graded k -algebras shows that the socle of $k \otimes_R E$ is $k \otimes_R E_c$. As $k \otimes_R \gamma$ is a homomorphism of graded k -algebras, when $k \otimes_R \gamma_c$ is injective so is $k \otimes_R \gamma$; now use (1.4.2). \square

The preceding description brings to light a connection between 1-closure for parameter ideals and Hochster's Canonical Element Conjecture, see [19].

1.6.2. Let R be a local ring. The following conditions are equivalent:

- (1) The Canonical Element Conjecture holds for R .
- (2) For each system of parameters \mathfrak{p} of R the map $\varphi: R \rightarrow R/(\mathfrak{p})$ is 1-closed.

Indeed, Roberts [21] has proved that the Canonical Element Conjecture holds for R if and only if for each free resolution F of $R/(\mathfrak{p})$ over R and each comparison morphism $\kappa: E \rightarrow F$, the induced map $(k \otimes_R E_c) \rightarrow \mathrm{H}_c(k \otimes_R F)$ is injective; another proof of his result is given by Huneke and Koh [17, (1.3)]. Thus, the desired equivalence is contained in (1.4.2) and Lemma (1.6.1).

The hypothesis on R in the next theorem reflects the use in its proof of a result of Bruns [10], which in turn relies on the Improved New Intersection Theorem.

Theorem 1.6.3. *Let $\varphi: R \rightarrow S$ be a 1-closed homomorphism and assume R contains a field as a subring. If $\mathrm{pd}_R S$ is finite, then φ is complete intersection.*

Proof. Set $I = \mathrm{Ker} \varphi$ and $c = \varepsilon_2(\varphi)$. The Koszul complex $E = G^{(1)}$ on a minimal generating set of I yields an injection $\kappa: k \otimes_R E \rightarrow \mathrm{Tor}^R(k, S)$, see (1.4.1).

Since $\mathrm{pd}_R S$ is finite, [10, Lemma 2] yields $\kappa_i = 0$ for $i > \mathrm{height} I$. By construction one has $\mathrm{rank}_R E_1 = c$, and this implies $c \leq \mathrm{height} I$. The reverse inequality always holds, due to the Principal Ideal Theorem, hence one gets $\mathrm{height} I = c$. On the other hand, $\mathrm{pd}_R S < \infty$ implies that $\mathrm{height} I$ equals the maximal length of an R -regular sequence in I , see [3, (2.5)]. We conclude that I can be generated by an R -regular sequence, as desired. \square

2. BOUNDS ON HOMOLOGY

The main result of this section is a condition for a 2-closed homomorphism to be c.i. When φ admits a section and $M = S$ it specializes to a result of Rodicio, [22, Theorem 1]. The reason for dealing with complexes, rather than just with modules, will become apparent in the proof of Theorem (3.1).

Theorem 2.1. *Let $\varphi: R \rightarrow S$ be a 2-closed local homomorphism and M a complex of S -modules with $\mathrm{H}(M)$ degreewise finite and bounded below.*

If there exist integers $t, u \geq \inf \mathrm{H}(M)$ of different parity, such that

$$\mathrm{Tor}_t^R(S, M) = 0 = \mathrm{Tor}_u^R(S, M)$$

then the homomorphism φ is complete intersection.

We comment on notions and notation appearing in the theorem and its proof.

2.2. For definitions of Tor and Ext for complexes we refer to [25]. When their arguments are modules (modules are always identified with complexes concentrated

in degree 0) these are the classical derived functors. We set

$$\begin{aligned}\inf H(M) &= \inf\{n \mid H_n(M) \neq 0\} \\ \sup H(M) &= \sup\{n \mid H_n(M) \neq 0\}\end{aligned}$$

When $\inf H(M)$ (respectively, $\sup H(M)$) is finite we say that $H(M)$ is *bounded below* (respectively, *above*). If M is bounded on either side, then $H(M) \neq 0$, because $H(M) = 0$ is equivalent to $\inf H(M) = \infty$, and also to $\sup H(M) = -\infty$.

For each integer j a complex $\Sigma^j M$ is defined by

$$\Sigma^j(M)_n = M_{n-j} \quad \text{and} \quad \partial_n^{\Sigma^j M} = (-1)^{|j|} \partial_{n-j}^M$$

Morphisms of complexes are chain maps of degree 0. A *quasiisomorphism* is a morphism that induces isomorphisms in homology in all degrees; we tag quasiisomorphisms with the symbol \simeq , and isomorphisms with \cong .

We deduce Theorem (2.1) from the following, much stronger, result.

Theorem 2.3. *Let $\varphi: R \rightarrow S$ be a 2-closed local homomorphism, set $c = \varepsilon_2(\varphi)$ and $d = \varepsilon_3(\varphi)$. If M is a complex of S -modules with $H(M)$ degreewise finite and bounded below, then for $i = \inf H(M)$ and $m = \nu_S(H_i(M))$ one has inequalities*

$$\begin{aligned}\nu_S(\mathrm{Tor}_{n+i}^R(S, M)) &\geq m \cdot \binom{c}{n} \quad \text{for } 0 \leq n \leq c \\ \nu_S(\mathrm{Tor}_{2n+i+c}^R(S, M)) &\geq m \cdot \binom{n+d-1}{d-1} \quad \text{for } 1 \leq n\end{aligned}$$

The proof uses a general lemma in homological algebra, presented below.

2.4. Let T be a covariant additive functor from the category of complexes of S -modules to the category of graded S -modules; for each complex M of S -modules we write $T_n(M)$ for the component in degree n of the graded S -module $T(M)$. Assume, furthermore, that T has the following properties:

- (a) T preserves quasiisomorphisms.
- (b) T commutes with shifts: $T_n(\Sigma^j M) = T_{n-j}(M)$ for each $n \in \mathbb{Z}$.
- (c) $T(\mu_s^M) = \mu_s^{T(M)}$ for each $s \in S$, where μ_s denotes multiplication by s .

For the maximal ideal \mathfrak{n} of S property (c) implies:

2.4.1. One has $\mathfrak{n} \cdot T(k) = 0$, so $T(k)$ is naturally a graded k -vector space.

Lemma 2.4.2. *Let $\varphi: R \rightarrow S$ be a surjective homomorphism and $\epsilon: S \rightarrow k$ the canonical surjection. If M is a complex of S -modules as in Theorem (2.3), then*

$$\nu_S(T_{n+i}(M)) \geq m \cdot \mathrm{rank}_k \mathrm{Im}(T_n(\epsilon))$$

Proof. First we simplify M . The inclusion into M of the subcomplex

$$M' := \cdots \longrightarrow M_{i+2} \xrightarrow{\partial_{i+2}} M_{i+1} \longrightarrow \mathrm{Ker}(\partial_i) \longrightarrow 0$$

is a quasiisomorphism. By (a) one has $T_n(M) \cong T_n(M')$, so we may assume $M = M'$. Set $H = H_i(M)$, choose a surjection $H \rightarrow k^m$ and let π denote the composition

$M \rightarrow \Sigma^i H \rightarrow \Sigma^i k^m$ of morphisms of complexes of S -modules. Lifting $\Sigma^i \epsilon^m$ over π to a morphism $\rho: \Sigma^i S^m \rightarrow M$, we get a commutative diagram

$$\begin{array}{ccccc}
 & & T_{n+i}(\Sigma^i S^m) & \xrightarrow{\cong} & T_{n+i}(\Sigma^i S)^m \\
 & \swarrow T_{n+i}(\rho) & \downarrow T_{n+i}(\Sigma^i \epsilon^m) & & \downarrow T_{n+i}(\Sigma^i \epsilon)^m \\
 T_{n+i}(M) & & & & \\
 & \searrow T_{n+i}(\pi) & T_{n+i}(\Sigma^i k^m) & \xrightarrow{\cong} & T_{n+i}(\Sigma^i k)^m
 \end{array}$$

of homomorphism of S -modules. We can now write the relations below

$$\begin{aligned}
 \nu_S(T_{n+i}(M)) &\geq \text{rank}_k \text{Im}(T_{n+i}(\pi)) \\
 &\geq \text{rank}_k \text{Im}(T_{n+i}(\Sigma^i \epsilon^m)) \\
 &= m \cdot \text{rank}_k \text{Im}(T_{n+i}(\Sigma^i \epsilon)) \\
 &= m \cdot \text{rank}_k \text{Im}(T_n(\epsilon))
 \end{aligned}$$

by using consecutively the following facts: the maximal ideal of S annihilates $T_{n+i}(\Sigma^i k)$; the diagram commutes; $T_{n+i}(\Sigma^i k)$ is isomorphic to $T_n(k)$. \square

We need an explicit description of a subcomplex of an acyclic closure G of φ .

2.5. In the notation of (1.1), each R -module $G_n^{(2)}$ has a basis

$$\left\{ x_i y_j \mid \mathbf{i} \subseteq [1, c], \mathbf{j} = (j_1, \dots, j_d) \in \mathbb{N}^d, \text{card } \mathbf{i} + 2 \sum_{h=1}^d j_h = n \right\}$$

where $[1, c] = \{1, \dots, c\}$. Let a_1, \dots, a_c be a minimal set of generators of $\text{Ker } \varphi$ and E the Koszul complex on it. The differential of $G^{(2)}$ then has the form

$$\partial(x_i y_j) = \sum_{i \in \mathbf{i}} \pm a_i x_{i \setminus \{i\}} y_j + \sum_{i \in [1, c]} \sum_{j=1}^d \pm b_{ij} x_{i \cup \{i\}} y_{j - \mathbf{e}_j}$$

where $\mathbf{e}_j \in \mathbb{N}^d$ is the j th unit vector, and

$$z_j = \sum_{i=1}^c b_{ij} x_i \in G_1^{(2)} \quad \text{for } j = 1, \dots, c$$

are cycles whose homology classes minimally generate $H_1(E)$.

All the coefficients a_i and b_{ij} are in \mathfrak{m} : this is clear for the a_i ; as they form a minimal set of generators the relation $0 = \partial(z_j) = \sum_{i=1}^c b_{ij} a_i$ implies $b_{ij} \in \mathfrak{m}$.

Proof of Theorem (2.3). Let $\epsilon: S \rightarrow k$ be the canonical surjection. Lemma (2.4.2) applied to the functor T defined by $T_n(M) = \text{Tor}_n^R(S, M)$ yields

$$\nu_S(\text{Tor}_{n+i}^R(S, M)) \geq m \cdot \text{rank}_k \text{Im}(\text{Tor}_n^R(S, \epsilon))$$

Next we estimate the rank on the right hand side. Let $G^{(2)}$ be the second stage in an acyclic closure of φ , F a free resolution of S over R , and let $\gamma: G^{(2)} \rightarrow F$ be a

comparison morphism. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n(G^{(2)} \otimes_R S) & \xrightarrow{\mathrm{H}_n(\gamma \otimes_R S)} & \mathrm{H}_n(F \otimes_R S) & \xlongequal{\quad} & \mathrm{Tor}_n^R(S, S) \\
\mathrm{H}_n(G^{(2)} \otimes_R \epsilon) \downarrow & & \downarrow \mathrm{H}_n(F \otimes_R \epsilon) & & \downarrow \mathrm{Tor}_n^R(S, \epsilon) \\
\mathrm{H}_n(G^{(2)} \otimes_R k) & \xrightarrow{\mathrm{H}_n(\gamma \otimes_R k)} & \mathrm{H}_n(F \otimes_R k) & \xlongequal{\quad} & \mathrm{Tor}_n^R(S, k)
\end{array}$$

As $\mathrm{H}_n(\gamma \otimes_R k)$ is injective by (1.4.1), for each n we get

$$\mathrm{rank}_k(\mathrm{Im} \mathrm{Tor}_n^R(S, \epsilon)) \geq \mathrm{rank}_k \mathrm{Im}(\mathrm{H}_n(G^{(2)} \otimes_R \epsilon))$$

From the description of $G^{(2)}$ in (2.5) one sees that the graded submodule

$$Z = \bigoplus_{i \subseteq [1, c]} S(x_i y_0 \otimes 1) \oplus \bigoplus_{j \in \mathbb{N}^d \setminus 0} S(x_{[1, c]} y_j \otimes 1) \subseteq G^{(2)} \otimes_R S$$

consists of cycles and the differential of $G^{(2)} \otimes_R k$ is trivial; thus the composition

$$Z \otimes_S k \longrightarrow k \otimes_S \mathrm{H}(G^{(2)} \otimes_R S) \longrightarrow \mathrm{H}(G^{(2)} \otimes_R k) = G^{(2)} \otimes_R k$$

is injective. Counting ranks over k one obtains inequalities

$$\begin{aligned}
\mathrm{rank}_k \mathrm{Im}(\mathrm{H}_n(G^{(2)} \otimes_R \epsilon)) &\geq \binom{c}{n} && \text{for } 0 \leq n \leq c \\
\mathrm{rank}_k \mathrm{Im}(\mathrm{H}_{2n+c}(G^{(2)} \otimes_R \epsilon)) &\geq \binom{n+d-1}{n} && \text{for } 1 \leq n
\end{aligned}$$

To get the desired result, concatenate the (in)equalities established above. \square

Proof of Theorem (2.1). By hypothesis, one has $\mathrm{Tor}_t^R(S, M) = 0 = \mathrm{Tor}_u^R(S, M)$ for integers t, u satisfying $t, u \geq \inf(\mathrm{H}(M)) = i > -\infty$ and $t \not\equiv u \pmod{2}$. The first inequality established in Theorem (2.3) implies $t, u > i + c$ for $c = \varepsilon_2(\varphi)$. For $d = \varepsilon_3(\varphi)$ its second inequality in the theorem then yields $\binom{n+d-1}{d-1} = 0$ for some $n \geq 1$, forcing $d = 0$. Thus, φ is complete intersection by (1.2.3). \square

3. VANISHING OF COHOMOLOGY

In this section we provide cohomological criteria for a 2-closed homomorphism to be c.i. This uses a notion of *depth* of a complex M , defined by

$$\mathrm{depth}_S M = \inf\{n \in \mathbb{Z} \mid \mathrm{Ext}_S^n(k, M) \neq 0\}$$

This is the classical concept when M is a finite S -module.

Theorem 3.1. *Let $\varphi: R \rightarrow S$ be a 2-closed homomorphism and M a complex of S -modules with $\mathrm{H}(M)$ degreewise finite and bounded above.*

If there exist integers $t, u \geq \mathrm{depth}_S M - \dim S$, of different parity, such that

$$\mathrm{Ext}_R^{t+n}(S, M) = 0 = \mathrm{Ext}_R^{u+n}(S, M) \quad \text{for } 0 \leq n \leq \max\{\dim_S \mathrm{H}_n(M) \mid n \in \mathbb{Z}\}$$

then the homomorphism φ is complete intersection.

Remark. As one always has $\dim S - \mathrm{depth}_S M \geq \sup \mathrm{H}(M)$, see [12, (2.11.3)], the bound on t, u in the theorem may be replaced by $t, u \geq -\sup \mathrm{H}(M)$.

Theorem (3.1) is a cohomological counterpart to Theorem (2.1), which provides a main ingredient in its proof. Another component is the use of properties of dualizing complexes, reviewed below; we refer to Hartshorne [15] for details.

3.2. Dualizing complexes. A *dualizing complex* for (S, \mathfrak{n}, k) is a complex

$$D = 0 \rightarrow D_0 \rightarrow D_{-1} \rightarrow \cdots \rightarrow D_{-\dim S} \rightarrow 0$$

of injective modules with $H(D)$ degreewise finite and $\mathrm{Hom}_S(k, D) \simeq \Sigma^{-\dim S} k$.

Up to a quasiisomorphism of complexes, S has at most one dualizing complex. Such a complex exists when the local ring S is complete.

For each complex of S -modules M we set $M^\dagger = \mathrm{Hom}_S(M, D)$.

3.2.1. If $H(M)$ is degreewise finite, then so is $H(M^\dagger)$.

3.2.2. If $H(M)$ is bounded on one side, then $H(M^\dagger)$ is bounded on the other.

Lemma 3.2.3. *If $H(M)$ is degreewise finite and bounded above, then*

$$\inf H(M^\dagger) = \mathrm{depth}_S M - \dim S$$

Proof. The complex $H(M^\dagger)$ is degreewise finite and bounded below, see (3.2). This implies, the first equality below; the second one holds by definition:

$$\inf H(M^\dagger) = \inf H(k \otimes_S^{\mathbf{L}} M^\dagger) = \inf H(k \otimes_S^{\mathbf{L}} \mathrm{Hom}_S(M, D))$$

To compute the right hand side we use a sequence of quasiisomorphisms:

$$\begin{aligned} k \otimes_S^{\mathbf{L}} \mathrm{Hom}_S(M, D) &\simeq \mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(k, M), D) \\ &\simeq \mathrm{Hom}_S(\mathrm{Ext}_S(k, M), D) \\ &\cong \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), \mathrm{Hom}_S(k, D)) \\ &\simeq \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), \Sigma^{-\dim S} k) \\ &\simeq \Sigma^{-\dim S} \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k) \end{aligned}$$

The first one holds because k has a resolution by finite free S -modules and D is a bounded complex of injectives. For the second, note that $\mathbf{R}\mathrm{Hom}_S(k, M)$ can be represented by a complex of S -modules annihilated by \mathfrak{n} , so it is quasiisomorphic to its own homology, namely, $\mathrm{Ext}_S(k, M)$. The third one holds because $\mathrm{Ext}_S(k, M)$ is a direct sum of copies of shifts of k . The fourth quasiisomorphism is induced by $\mathrm{Hom}_S(k, D) \simeq \Sigma^{-\dim S} k$; see (3.2). The last one is standard.

We now finish the computation of $\inf H(M^\dagger)$ as follows:

$$\begin{aligned} \inf H(M^\dagger) &= \inf (\Sigma^{-\dim S} \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k)) \\ &= \inf \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k) - \dim S \\ &= \mathrm{depth}_S M - \dim S \end{aligned} \quad \square$$

3.2.4. For every finite S -module N one has

$$\mathrm{Ext}_S^n(N, D) = 0, \quad \text{unless } \dim S - \dim_S N \leq n \leq \dim S - \mathrm{depth}_S N$$

3.3. The *support* of a complex M is defined to be the set

$$\mathrm{Supp}_S M = \{\mathfrak{q} \in \mathrm{Spec} S \mid H(M_{\mathfrak{q}}) = 0\}$$

Let $\dim \mathrm{Supp}_S M$ denote the dimension of space $\mathrm{Supp}_S M$ in the Zariski topology on $\mathrm{Spec} S$. It is not hard to see that if $H(M)$ is degreewise finite, then

$$\dim \mathrm{Supp}_S M = \max\{\dim_S H_n(M) \mid n \in \mathbb{Z}\}$$

Proof of Theorem (3.1). By (1.4.3), the map $\widehat{\varphi}: \widehat{R} \rightarrow \widehat{S}$ of maximal-ideal-adic completions induced by φ is 2-closed. For $\widehat{M} = M \otimes_S \widehat{S}$ and each $n \in \mathbb{Z}$ one has

$$\mathrm{H}_n(\widehat{M}) \cong \mathrm{H}_n(M) \otimes_R \widehat{R} \quad \text{and} \quad \mathrm{Ext}_R^n(\widehat{S}, \widehat{M}) \cong \mathrm{Ext}_R^n(S, M) \otimes_R \widehat{R}$$

where the first one is due to the flatness of \widehat{R} over R , while the second uses, in addition, that S is finite over R and that $\mathrm{H}(M)$ is bounded above. In particular, one has $\dim_{\widehat{S}} \mathrm{H}_n(\widehat{M}) = \dim_S \mathrm{H}_n(M)$ for each $n \in \mathbb{Z}$. Thus, the hypotheses of the theorem do not change when R, S, M are replaced by $\widehat{R}, \widehat{S}, \widehat{M}$, respectively. Furthermore, if $\widehat{\varphi}$ is c.i., then so is φ . Thus, we may assume that the ring S is complete, and hence that it has a dualizing complex D . Set $m = \max\{\dim_S \mathrm{H}_n(M) \mid n \in \mathbb{Z}\}$.

As D is a bounded complex of injectives, there is a natural quasiisomorphism

$$(*) \quad \mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S, M), D) \simeq S \otimes_R^{\mathbf{L}} \mathrm{Hom}_S(M, D)$$

The composition of the factors on the left gives rise to a spectral sequence with

$${}_2\mathrm{E}^{p,q} = \mathrm{Ext}_S^{-p}(\mathrm{Ext}_R^q(S, M), D) \quad \text{and} \quad {}_r\mathrm{d}^{p,q}: {}_r\mathrm{E}^{p,q} \longrightarrow {}_r\mathrm{E}^{p-r, q+r-1}$$

As the R -module S is finite, one has $\mathrm{Ext}_R^q(S, M)_{\mathfrak{q}} \cong \mathrm{Ext}_R^q(S, M_{\mathfrak{q}})$ for each $\mathfrak{q} \in \mathrm{Spec} S$, so $\mathrm{Supp}_S \mathrm{Ext}_R^q(S, M) \subseteq \mathrm{Supp}_S M$, for each $q \in \mathbb{Z}$. Thus, one gets

$$\dim_{S_{\mathfrak{q}}}(\mathrm{Ext}_R^q(S, M)_{\mathfrak{q}}) \leq \dim \mathrm{Supp}_S M = m$$

where the equality comes from (3.3). Now (3.2.4) yields

$${}_2\mathrm{E}^{p,q} = 0 \quad \text{for} \quad p \notin [-\dim S, -\dim S + m]$$

so the sequence converges. Formula (*) shows that its abutment is equal to

$$\mathrm{H}(S \otimes_R^{\mathbf{L}} \mathrm{Hom}_S(M, D)) = \mathrm{Tor}_{p+q}^R(S, M^\dagger)$$

On the other hand, our hypothesis entails ${}_2\mathrm{E}^{p,q} = 0$ for

$$t \leq q \leq t + m \quad \text{and} \quad u \leq q \leq u + m$$

As a consequence, one obtains equalities

$${}_2\mathrm{E}^{p,q} = 0 \quad \text{whenever} \quad p + q = t \quad \text{or} \quad p + q = u$$

They imply ${}_{\infty}\mathrm{E}^{p,q} = 0$ if $p + q = t$ or $p + q = u$, so convergence yields

$$\mathrm{Tor}_t^R(S, M^\dagger) = 0 = \mathrm{Tor}_u^R(S, M^\dagger)$$

In view of Lemma (3.2.3) and our hypothesis, the complex M^\dagger satisfies

$$\inf \mathrm{H}(M^\dagger) = \mathrm{depth}_S M^\dagger - \dim S \leq \min\{t, u\}$$

Now Theorem (2.1), applied to M^\dagger , shows that φ is complete intersection. \square

4. (CO)HOMOLOGY OF ALGEBRA RETRACTS

Let $\varphi: R \rightarrow S$ be a homomorphism of noetherian rings.

A *section* of φ is a homomorphism of rings $\psi: S \rightarrow R$ such that $\psi \circ \varphi = \mathrm{id}^S$; when such a homomorphism exists S is said to be an *algebra retract* of R . Another way to describe this situation is to say that R is a *supplemented algebra* over S . The study of homological and cohomological properties of supplemented algebras is a central topic in the classical literature on homological algebra.

Each $\mathfrak{q} \in \mathrm{Spec} S$ defines a local homomorphism $\varphi_{\mathfrak{q}}: R_{\varphi^{-1}(\mathfrak{q})} \rightarrow S_{\mathfrak{q}}$. If ψ is a section of φ , the local homomorphism $\psi_{\mathfrak{p}}$, where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, is a section of $\varphi_{\mathfrak{q}}$. In particular, the homomorphism $\varphi_{\mathfrak{q}}$ is 2-closed; see (1.3.3).

Next we establish global versions of results from the preceding sections. To this end we recall the construction of certain canonical homomorphisms.

4.1. With $I = \text{Ker}(\varphi)$, one has a canonical of S -modules isomorphism

$$I/I^2 \cong \text{Tor}_1^R(S, S)$$

The graded S -module $\text{Tor}_*^R(S, S)$ has a natural structure of a strictly commutative graded S -algebra, see [25, (2.7.8)], so there is a homomorphism of graded S -algebras: $\lambda^S: \wedge_S(I/I^2) \rightarrow \text{Tor}^R(S, S)$. Define λ^M to be the composition

$$\wedge_S(I/I^2) \otimes_S \mathbf{H}(M) \xrightarrow{\lambda^S \otimes_S \mathbf{H}(M)} \text{Tor}^R(S, S) \otimes_S \mathbf{H}(M) \longrightarrow \text{Tor}^R(S, M)$$

where the second arrow is a Künneth map. Let λ_M denote the composition

$$\text{Ext}_R(S, M) \longrightarrow \text{Hom}_S(\text{Tor}^R(S, S), \mathbf{H}(M)) \longrightarrow \text{Hom}_S(\wedge_S(I/I^2), \mathbf{H}(M))$$

where the first arrow is a Künneth map, and the second is $\text{Hom}_S(\lambda^S, \mathbf{H}(M))$

Theorem 4.2. *Let $\varphi: R \rightarrow S$ be a homomorphism of rings that admits a section, and let M be a complex of S -modules with $\mathbf{H}(M)$ finite. Set $I = \text{Ker}(\varphi)$.*

For each prime ideal $\mathfrak{q} \in \text{Supp}_S M$ the following conditions are equivalent.

- (i) *The homomorphism $\varphi_{\mathfrak{q}}$ is complete intersection.*
- (ii_{*}) *The map $(\lambda^M)_{\mathfrak{q}}$ is bijective and the $S_{\mathfrak{q}}$ -module $(I/I^2)_{\mathfrak{q}}$ is projective.*
- (iii_{*}) *For integers $t, u \geq \inf \mathbf{H}(M)_{\mathfrak{q}}$ of different parity one has*

$$\text{Tor}_t^R(S, M)_{\mathfrak{q}} = 0 = \text{Tor}_u^R(S, M)_{\mathfrak{q}}$$

- (ii^{*}) *The map $(\lambda_M)_{\mathfrak{q}}$ is bijective.*

- (iii^{*}) *For integers $t, u \geq \text{depth}_{S_{\mathfrak{q}}} M_{\mathfrak{q}} - \dim S_{\mathfrak{q}}$ of different parity one has*

$$\text{Ext}_R^{t+i}(S, M)_{\mathfrak{q}} = 0 = \text{Ext}_R^{u+i}(S, M)_{\mathfrak{q}} \quad \text{for } i = 0, \dots, \dim \text{Supp}_{S_{\mathfrak{q}}} M_{\mathfrak{q}}$$

Proof. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. The R -module S is finite, so for each $n \in \mathbb{Z}$ one has

$$\text{Tor}_n^R(S, M)_{\mathfrak{q}} \cong \text{Tor}_n^{R_{\mathfrak{p}}}(S_{\mathfrak{q}}, M_{\mathfrak{q}}) \quad \text{and} \quad \text{Ext}_R^n(S, M)_{\mathfrak{q}} \cong \text{Ext}_{R_{\mathfrak{p}}}^n(S_{\mathfrak{q}}, M_{\mathfrak{q}})$$

Therefore, each of the conditions listed above is local; moreover, any section of φ localizes to a section of $\varphi_{\mathfrak{q}}$. Thus, changing notation we may assume that φ is a local homomorphism and that \mathfrak{q} is the maximal ideal of S .

(i) \implies (ii_{*}) and (ii^{*}). Let E be the Koszul complex on a minimal generating set of I . It satisfies $\partial(E) \subseteq IE$ and is a free resolution of S , as φ is c.i. We get

$$\text{Tor}^R(S, S) \cong \mathbf{H}(E \otimes_R S) = E \otimes_R S$$

so the S -module $\text{Tor}_1^R(S, S)$ is free and λ^S is bijective. Thus, $\text{Tor}_n^R(S, S)$ is finite free and vanishes for $n < 0$ or $n > \nu_S(I)$, so the Künneth homomorphisms

$$\begin{aligned} \text{Tor}^R(S, S) \otimes_S \mathbf{H}(M) &\longrightarrow \text{Tor}^R(S, M) \\ \text{Ext}_R(S, M) &\longrightarrow \text{Hom}_S(\text{Tor}^R(S, S), \mathbf{H}(M)) \end{aligned}$$

are bijective. The definitions of λ^M and λ_M show that they are bijective as well.

(ii_{*}) \implies (iii_{*}), and (ii^{*}) \implies (iii^{*}). These implications are clear because the S -module I/I^2 is finite and $\mathbf{H}(M)$ is bounded.

(iii_{*}) or (iii^{*}) \implies (i). As φ is closed by (1.3.3), Theorem (2.1), respectively, Theorem (3.1), shows that condition (iii_{*}), respectively, (iii^{*}), implies φ is c.i. \square

5. HOCHSCHILD (CO)HOMOLOGY

Finally, we return to the subject in the title of this article. First we recall a classical interpretation of the functors in question; see e.g. [25, (9.1.5)].

5.1. Let $\eta: K \rightarrow S$ be a homomorphism of rings and let $\varphi^S: S \otimes_K S \rightarrow S$ be the homomorphism of rings given by $\varphi^S(s \otimes_K s') = ss'$.

If the K -module S is flat, then for each $n \in \mathbb{Z}$ one has

$$\mathrm{HH}_n(S|K; M) = \mathrm{Tor}_n^{S \otimes_K S}(S, M)$$

Note that I/I^2 , where $I = \mathrm{Ker}(\varphi^S)$, is a standard realization of the module of differentials $\Omega_{S|K}$. The map λ^M from (4.1) yields S -linear maps

$$\lambda_n^M: \wedge_S^n \Omega_{S|K} \otimes_S M \longrightarrow \mathrm{HH}_n(S|K; M)$$

If S is projective as a K -module, then also

$$\mathrm{HH}^n(S|K; M) = \mathrm{Ext}_{S \otimes_K S}^n(S, M)$$

so in this context the homomorphism λ_M from (4.1) reads

$$\lambda_M^n: \mathrm{HH}^n(S|K; M) \longrightarrow \mathrm{Hom}_S(\wedge_S^n \Omega_{S|K}, M)$$

The maps above are the homomorphisms that appear in the introduction. For the proof of the theorem stated there we need a characterization of smoothness proved by André [1, Proposition C], using André-Quillen homology. A short version of his argument may be found in [4, (1.1)].

5.2. A flat algebra S essentially of finite type over a noetherian ring K is smooth if and only if the homomorphism $(\varphi^S)_{\mathfrak{p}}$ is c.i., for each $\mathfrak{p} \in \mathrm{Spec} S$.

Proof of the Main Theorem. Let $\varphi^S: S \otimes_K S \rightarrow S$ be the product map. We claim that, for a given $\mathfrak{q} \in \mathrm{Spec} S$, condition (i): the K -algebra $S_{\mathfrak{q}}$ is smooth, is equivalent to: (i') the homomorphism $(\varphi^S)_{\mathfrak{q}}: (S \otimes_K S)_{(\varphi^S)^{-1}(\mathfrak{q})} \rightarrow S_{\mathfrak{q}}$ is c.i.

Indeed, $(\varphi^S)_{\mathfrak{q}}$ is surjective, so it is c.i. if and only if $(\varphi^S)_{\mathfrak{p}}$ is c.i. for each $\mathfrak{p} \subseteq \mathfrak{q}$. However, the local homomorphisms $(\varphi^S)_{\mathfrak{p}}$ and $(\varphi^{S_{\mathfrak{q}}})_{\mathfrak{p}}$ coincide, and the latter is c.i. for each \mathfrak{p} precisely when the K -algebra $S_{\mathfrak{q}}$ is smooth, by (5.2).

Given this translation and the identifications in (5.1), the desired result is contained in Theorem (4.2), for $s \mapsto 1 \otimes s$ gives a section $S \rightarrow S \otimes_K S$ of φ^S . \square

The example below shows that condition (ii_{*}) in the Main Theorem cannot be weakened in general. We do not know whether the conclusion of the theorem still holds if the vanishing intervals in condition (iii_{*}) are shortened.

Example 5.3. Let $S = \mathbb{Z}[\sqrt{2}]$. The Hochschild homology of S over \mathbb{Z} is

$$\mathrm{HH}_n(S|\mathbb{Z}; S) = \begin{cases} S & \text{for } n = 0 \\ S/(2\sqrt{2}) & \text{for odd } n \geq 1 \\ 0 & \text{for even } n \geq 2 \end{cases}$$

while the Hochschild cohomology of S over \mathbb{Z} is given by

$$\mathrm{HH}^n(S|\mathbb{Z}; S) = \begin{cases} S & \text{for } n = 0 \\ 0 & \text{for odd } n \geq 1 \\ S/(2\sqrt{2}) & \text{for even } n \geq 2 \end{cases}$$

Indeed, $\text{Ker}(S \otimes_{\mathbb{Z}} S \rightarrow S)$ is generated by $\sqrt{2} \otimes 1 - 1 \otimes \sqrt{2}$. A free resolution of S as a module over $S \otimes_{\mathbb{Z}} S$ is given by the complex F below:

$$\cdots \longrightarrow S \otimes_{\mathbb{Z}} S \xrightarrow{\sqrt{2} \otimes 1 - 1 \otimes \sqrt{2}} S \otimes_{\mathbb{Z}} S \xrightarrow{\sqrt{2} \otimes 1 + 1 \otimes \sqrt{2}} S \otimes_{\mathbb{Z}} S \xrightarrow{\sqrt{2} \otimes 1 - 1 \otimes \sqrt{2}} S \otimes_{\mathbb{Z}} S \longrightarrow 0$$

As S is finite free as a \mathbb{Z} -module, $\text{HH}_*(S|\mathbb{Z}; S)$ is the homology of the complex

$$F \otimes_{S \otimes_{\mathbb{Z}} S} S = \cdots \longrightarrow S \xrightarrow{2\sqrt{2}} S \xrightarrow{0} S \xrightarrow{2\sqrt{2}} S \xrightarrow{0} S \longrightarrow 0$$

and $\text{HH}^*(S|\mathbb{Z}; S)$ is the homology of the complex

$$\text{Hom}_{S \otimes_{\mathbb{Z}} S}(F, S) = 0 \longrightarrow S \xrightarrow{0} S \xrightarrow{2\sqrt{2}} S \xrightarrow{0} S \xrightarrow{2\sqrt{2}} S \longrightarrow \cdots$$

see (5.1). The desired expressions follow.

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