

# FROBENIUS POWERS OF COMPLETE INTERSECTIONS

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## INTRODUCTION

Let  $R$  be a commutative noetherian local ring of characteristic  $p > 0$ , and let  $\phi: R \rightarrow R$  be the Frobenius endomorphism,  $\phi(a) = a^p$ . Each iteration  $\phi^r$  defines on  $R$  a new structure of  $R$ -module, denoted  $\phi^r R$ , for which  $a \cdot b = a^{p^r} b$ .

In 1969 Kunz [7, (3.3)] observed that if  $R$  is regular, then  $\phi^r R$  is flat for all  $r \geq 0$ , and he discovered that if  $\phi^r R$  is flat for some  $r \geq 1$ , then  $R$  is regular. Regularity is equivalent to the finiteness of the projective dimension of the  $R$ -module  $k = R/\mathfrak{m}$ , where  $\mathfrak{m}$  is the maximal ideal of  $R$ , so Kunz's theorem connects the homological properties of  $k$  and those of  $\phi$ . To summarize further results along these lines, we let  $c(R)$  denote the least integer  $s$  such that  $(\mathbf{y} : \mathfrak{m}) \not\subseteq \mathfrak{m}^s$  for some maximal  $R$ -regular sequence  $\mathbf{y}$  (such an  $s$  exists by Krull's Intersection Theorem).

For a finitely generated  $R$ -module  $M$  the following conditions are equivalent.

- (i)  $M$  has finite projective dimension.
- (ii)  $\mathrm{Tor}_n^R(M, \phi^r R) = 0$  for all  $n, r \geq 1$ .
- (iii)  $\mathrm{Tor}_n^R(M, \phi^r R) = 0$  for all  $n \geq 1$  and infinitely many  $r$ .
- (iv)  $\mathrm{Tor}_n^R(M, \phi^r R) = 0$  for  $j \leq n \leq j + \mathrm{depth} R + 1$  where  $j, r$  are fixed integers satisfying  $j \geq 1$  and  $r > \log_p(c(R))$ .

The implication (i)  $\implies$  (ii) is a fundamental theorem of Peskine and Szpiro [10, (1.7)]. An early converse, (iii)  $\implies$  (i), was given by Herzog [4, (3.1)]. Recently, Koh and Lee proved (iv)  $\implies$  (i) in [6, (2.6)] (but stated a weaker result).

The local ring  $R$  is said to be *complete intersection* if in some (equivalently, in every) Cohen presentation of its  $\mathfrak{m}$ -adic completion as a homomorphic image of a regular local ring, the defining ideal is generated by a regular sequence. If  $R$  has this property, the length  $\ell_R M$  of  $M$  is finite, and its projective dimension  $\mathrm{pd}_R M$  is infinite, then  $\lim_{r \rightarrow \infty} (\ell_R(\mathrm{Tor}_n^R(M, \phi^r R)) p^{-r \dim R})$  exists by Seibert [11, Prop. 1], and is positive by Miller [9, (2.5)]; this partly sharpens Herzog's theorem.

Our main result links, qualitatively and quantitatively, the homology of Frobenius powers of a complete intersection and the homology of the residue field.

**Theorem.** *Let  $M$  be a module over a complete intersection local ring  $(R, \mathfrak{m}, k)$ .*

*If  $\mathrm{Tor}_j^R(M, \phi^r R) = 0$  for some fixed  $j, r \geq 1$  then  $\mathrm{Tor}_n^R(M, \phi^r R) = 0$  for all  $n \geq j$ ; if, furthermore,  $M$  is finitely generated, then  $\mathrm{pd}_R M < \infty$ .*

*If  $M$  has finite length and  $\mathrm{pd}_R M = \infty$ , then for every  $r \geq 1$  the limits*

$$\lim_{s \rightarrow \infty} \frac{\ell_R(\mathrm{Tor}_{2s}^R(M, \phi^r R))}{\ell_R(\mathrm{Tor}_{2s}^R(M, k))} \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{\ell_R(\mathrm{Tor}_{2s+1}^R(M, \phi^r R))}{\ell_R(\mathrm{Tor}_{2s+1}^R(M, k))}$$

*exist, are rational numbers, and at least one of them is positive.*

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It should be noted that none of the conclusions of the theorem requires  $R$  to be a complete intersection. While we do not know whether this hypothesis is necessary, it does play a major role in our proofs. We use it in Section 1 to show that  ${}^{\phi^r}R$  is rigid, refining techniques from [3], [9]. We invoke it again in Section 2 in order to apply results from [2], [8], [1], on the one hand to deduce finite projective dimension from rigidity, on the other to study the asymptotic behavior of lengths of Tor's.

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## 1. RIGIDITY

Throughout our discussion, different module structures on the same abelian group will be induced by various homomorphisms of commutative rings. We start by describing notation that will keep track of the module structure in use.

If  $\alpha: A \rightarrow B$  is a homomorphism of commutative rings, then  ${}^{\alpha}B$  denotes the  $A$ - $B$ -bimodule  $B$  with  $A$  acting through  $\alpha$  and  $B$  acting through  $\text{id}_B$ , that is,  $a \cdot b' = \alpha(a)b'$  and  $b \cdot b' = bb'$  for all  $a \in A$ ,  $b' \in {}^{\alpha}B$ ,  $b \in B$ . For each  $A$ -module  $M$  the tensor product  $M \otimes_A {}^{\alpha}B$  is a  $B$ -module:  $b \cdot (m \otimes b') = m \otimes (bb')$  for all  $b \in B$ ,  $m \in M$ ,  $b' \in {}^{\alpha}B$ . Using a projective resolution of  $M$  to compute Tor's, one endows  $\text{Tor}_n^A(M, {}^{\alpha}B)$ , for each  $n \geq 0$ , with a  $B$ -module structure that is natural in  $M$ .

We fix a prime number  $p$  and an integer  $r > 0$ , and set  $q = p^r$ . For any ring  $A$  of characteristic  $p$ , we use  $\varphi$  to denote the  $r$ 'th iteration of the Frobenius endomorphism:  $\varphi(a) = a^q$  for all  $a \in A$ . For every homomorphism  $\alpha: A \rightarrow B$  one has  $\alpha\varphi = \varphi\alpha$ . For a subset  $\mathbf{a}$  of  $A$ , we sometimes write  $\mathbf{a}^q$  instead of  $\varphi(\mathbf{a})$ .

From now on  $R$  denotes a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = R/\mathfrak{m}$ . We let  $\iota: R \rightarrow \widehat{R}$  denote the canonical map into the  $\mathfrak{m}$ -adic completion.

**Remark 1.** For each  $R$ -module  $M$  and all  $n \geq 0$  there are natural isomorphisms

$$\text{Tor}_n^R(M, {}^{\varphi}R) \otimes_R {}^{\iota}\widehat{R} \cong \text{Tor}_n^R(M, {}^{\iota\varphi}\widehat{R}) = \text{Tor}_n^R(M, {}^{\varphi\iota}\widehat{R}) \cong \text{Tor}_n^{\widehat{R}}(M \otimes_R {}^{\iota}\widehat{R}, {}^{\varphi}\widehat{R})$$

obtained by standard use of the flatness of  $\iota$ .

**Remark 2.** By Cohen's Structure Theorem we may assume  $\widehat{R}$  is a residue ring of a ring of formal power series  $Q = k[[t]]$  on indeterminates  $\mathbf{t} = t_1, \dots, t_e$ .

Let  $-'$  denote the functor  $(- \otimes_Q {}^{\varphi}Q)$  from the category of  $Q$ -modules into itself; it is exact by Kunz's theorem. On the category of  $\widehat{R}$ -modules the functors  $(- \otimes_Q \widehat{R})$  and  $(- \otimes_{\widehat{R}} {}^{\varphi}\widehat{R})$  are isomorphic, by associativity of tensor products.

We further assume  $\widehat{R} = Q/(\mathbf{x})$  for a  $Q$ -regular sequence  $\mathbf{x} = x_1, \dots, x_c$ .

The subquotients of the  $(\mathbf{x})$ -adic filtration of the  $Q$ -algebra  $S = Q/(\mathbf{x}^q)$  are free  $\widehat{R}$ -modules. Refining this filtration, we fix a filtration

$$0 = S_{q^c} \subset S_{q^{c-1}} \subset \dots \subset S_1 \subset S_0 = S$$

with subquotients isomorphic to  $\widehat{R}$ ; it produces exact sequences

$$(1_i) \quad 0 \longrightarrow S_{i+1} \xrightarrow{\tau_i} S_i \xrightarrow{\sigma_i} \widehat{R} \longrightarrow 0 \quad \text{for } i = 0, \dots, q^c - 1.$$

For each  $Q$ -module  $N$  and for  $i = 0, \dots, q^c - 1$ , set  $S_i(N) = \text{Ker}(N' \otimes_Q \tau_i)$ .

The idea for the proof of part (b) below comes from [3, (2.2)] and [9, (2.1)].

**Lemma 3.** *If  $R = \widehat{R}$ , then for each  $R$ -module  $M$  the following hold.*

- (a)  $S_i(M)$  is a homomorphic image of  $S_0(M)$  for  $i = 1, \dots, q^c - 1$ .
- (b)  $S_0(M) \cong \text{Tor}_1^R(M, {}^{\varphi}R)$ .

*Proof.* Applying  $\mathrm{Tor}^Q(-, {}^vR)$  to each sequence (1<sub>*i*</sub>) we obtain isomorphisms

$$(2_i) \quad S_i(M) \cong \mathrm{Coker}(\mathrm{Tor}_1^Q(M', \sigma_i)) \quad \text{for } i = 0, \dots, q^c - 1.$$

(a) As  $S_0 = S$ , for each exact sequence of  $S$ -modules (1<sub>*i*</sub>) there exists a map  $\pi_i: S_0 \rightarrow S_i$  with  $\sigma_0 = \sigma_i \pi_i$ . In view of (2<sub>*i*</sub>), it yields a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Tor}_1^Q(M', S_0) & \xrightarrow{\mathrm{Tor}_1^Q(M', \sigma_0)} & \mathrm{Tor}_1^Q(M', R) & \longrightarrow & S_0(M) & \longrightarrow & 0 \\ \mathrm{Tor}_1^Q(M', \pi_i) \downarrow & & \parallel & & \downarrow \varpi_i & & \\ \mathrm{Tor}_1^Q(M', S_i) & \xrightarrow{\mathrm{Tor}_1^Q(M', \sigma_i)} & \mathrm{Tor}_1^Q(M', R) & \longrightarrow & S_i(M) & \longrightarrow & 0 \end{array}$$

where the rows are exact, and so the homomorphism  $\varpi_i$  is surjective.

(b) Choose an exact sequence  $0 \rightarrow K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \rightarrow 0$  with a free  $R$ -module  $L$ , then apply  $(- \otimes_Q {}^vQ)$  to get an exact sequence of  $Q$ -modules

$$(3) \quad 0 \rightarrow K' \xrightarrow{\kappa'} L' \xrightarrow{\lambda'} M' \rightarrow 0$$

Writing  $L = G \otimes_Q R$  with a free  $Q$ -module  $G$ , we obtain a commutative diagram

$$\begin{array}{ccccc} G \otimes_Q \mathrm{Tor}_1^Q(S, S) & \xrightarrow{G \otimes_Q \mathrm{Tor}_1^Q(S, \sigma)} & G \otimes_Q \mathrm{Tor}_1^Q(S, R) & & \\ \cong \downarrow & & \downarrow \cong & & \\ \mathrm{Tor}_1^Q(L, R) \otimes_Q {}^vQ & \xrightarrow{\cong} & \mathrm{Tor}_1^Q(L', S) & \xrightarrow{\mathrm{Tor}_1^Q(L', \sigma)} & \mathrm{Tor}_1^Q(L', R) \\ \mathrm{Tor}_1^Q(\lambda, R) \otimes_Q {}^vQ \downarrow & & \downarrow \mathrm{Tor}_1^Q(\lambda', S) & & \downarrow \mathrm{Tor}_1^Q(\lambda', R) \\ \mathrm{Tor}_1^Q(M, R) \otimes_Q {}^vQ & \xrightarrow{\cong} & \mathrm{Tor}_1^Q(M', S) & \xrightarrow{\mathrm{Tor}_1^Q(M', \sigma)} & \mathrm{Tor}_1^Q(M', R) \end{array}$$

with isomorphisms due to the flatness of  $G$  and  ${}^vQ$  over  $Q$ , and the equality  $R' = S$ .

The Koszul complex  $K(\mathbf{x}, Q)$  is a free resolution of  $R$  over  $Q$ . For each  $R$ -module  $N$  the differential of the complex  $N \otimes_Q K(\mathbf{x}, Q)$  is trivial, so there is an isomorphism  $\mathrm{Tor}_1^Q(-, R) \cong (- \otimes_R R^c)$  of functors on the category of  $R$ -modules. In particular,  $\mathrm{Tor}_1^Q(\lambda, R)$  is surjective, hence so is  $\mathrm{Tor}_1^Q(\lambda', S)$ . Similarly, the Koszul complex  $K(\mathbf{x}^q, Q)$  resolves  $S$  over  $Q$ . The differential of  $K(\mathbf{x}^q, Q) \otimes_Q N$  is trivial for each  $S$ -module  $N$ , so there is an isomorphism  $\mathrm{Tor}_1^Q(S, -) \cong (S^c \otimes_S -)$  of functors on the category of  $S$ -modules. Thus,  $\mathrm{Tor}_1^Q(S, \sigma)$  is surjective, hence so is  $\mathrm{Tor}_1^Q(L', \sigma)$ .

Formula (2<sub>0</sub>) and the preceding computations yield isomorphisms

$$S_0(M) \cong \mathrm{Coker}(\mathrm{Tor}_1^Q(M', \sigma)) \cong \mathrm{Coker}(\mathrm{Tor}_1^Q(\lambda', R)).$$

The exact sequence (3) induces the top row of the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Tor}_1^Q(L', R) & \xrightarrow{\mathrm{Tor}_1^Q(\lambda', R)} & \mathrm{Tor}_1^Q(M', R) & \longrightarrow & K' \otimes_Q R & \xrightarrow{\lambda' \otimes_Q R} & L' \otimes_Q R \\ & & & & \cong \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Tor}_1^R(M, {}^vR) & \longrightarrow & K \otimes_R {}^vR & \xrightarrow{\lambda \otimes_R {}^vR} & L \otimes_R {}^vR \end{array}$$

with isomorphisms from Remark 2. It gives isomorphisms that finish the proof:

$$\mathrm{Coker}(\mathrm{Tor}_1^Q(\lambda', R)) \cong \mathrm{Ker}(\lambda' \otimes_Q R) \cong \mathrm{Ker}(\lambda \otimes_R {}^vR) \cong \mathrm{Tor}_1^R(M, {}^vR). \quad \square$$

*Proof of Theorem. Part I.* We assume that  $M$  is an  $R$ -module and there exists a  $j \geq 1$  such that  $\mathrm{Tor}_j^R(M, {}^\varphi R) = 0$ , and prove that  $\mathrm{Tor}_n^R(M, {}^\varphi R) = 0$  for all  $n \geq j$ .

In view of Remark 1 and the faithful flatness of  $\iota: R \rightarrow \widehat{R}$ , we may assume that  $R$  is complete. Obvious inductive considerations show that it suffices to establish the vanishing of  $\mathrm{Tor}_{j+1}^R(M, {}^\varphi R)$ . Replacing  $M$  by a  $(j-1)$ st syzygy, and adjusting notation, we may change our hypothesis to read  $\mathrm{Tor}_1^R(M, {}^\varphi R) = 0$ . Thus, the proposition will be proved once we show that this implies  $\mathrm{Tor}_2^R(M, {}^\varphi R) = 0$ .

The exact sequences (1 <sub>$i$</sub> ) and (3), and Remark 2, yield commutative diagrams

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 0 & \longrightarrow & S_i(K) & \longrightarrow & K' \otimes_Q S_{i+1} & \xrightarrow{K' \otimes_Q \tau_i} & K' \otimes_Q S_i & \longrightarrow & K \otimes_R {}^\varphi R & \longrightarrow & 0 \\
 & & & & \downarrow \kappa' \otimes_Q S_{i+1} & & \downarrow \kappa' \otimes_Q S_i & & \downarrow & & \\
 & & 0 & \longrightarrow & L' \otimes_Q S_{i+1} & \longrightarrow & L' \otimes_Q S_i & \longrightarrow & L \otimes_R {}^\varphi R & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M' \otimes_Q S_{i+1} & \longrightarrow & M' \otimes_Q S_i & \longrightarrow & M \otimes_R {}^\varphi R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 
 \end{array}$$

for  $i = 0, \dots, q^c - 1$ . The rows are exact by Lemma 3. The columns are exact due to right exactness of tensor products and, for the rightmost one, to our hypothesis.

By decreasing induction on  $i$  we prove the labeled maps are injective. If  $i = q^c - 1$ , then  $S_{i+1} = 0$ , so all modules in the left hand column are trivial, and our assertion is clear. If  $0 \leq i < q^c - 1$ , then  $\kappa' \otimes_Q S_{i+1}$  is injective by the induction hypothesis. Applying the Snake Lemma to the two top rows we see that  $\kappa' \otimes_Q S_i$  is injective, then applying it to the two columns on the left we conclude that  $K' \otimes_Q \tau_i$  is injective.

The injectivity of  $K' \otimes_Q \tau_0$  yields  $S_0(K) = 0$ . Lemma 3 shows that  $\mathrm{Tor}_1^R(K, {}^\varphi R)$  vanishes. This module is isomorphic to  $\mathrm{Tor}_2^R(M, {}^\varphi R)$ , so we are done.  $\square$

## 2. COMPLEXITY

Let  $(S, \mathfrak{n}, k)$  be a local complete intersection, not necessarily of positive characteristic, and set  $\mathrm{codim} S = \ell_S(\mathfrak{n}/\mathfrak{n}^2) - \dim S$ . Following [1], we say that a pair  $(K, L)$  of finitely generated  $S$ -modules has *complexity*  $d$ , and write  $\mathrm{cx}_S(K, L) = d$ , if  $d$  is the least non-negative integer for which there exists a  $\beta \in \mathbb{R}$  such that

$$\ell_S(\mathrm{Ext}_S^n(K, L) \otimes_S k) \leq \beta n^{d-1} \quad \text{for all } n \gg 0.$$

The number  $\mathrm{cx}_S(K, k)$  is called the complexity of  $K$  and is denoted  $\mathrm{cx}_S K$ . It measures the polynomial rate of growth of a minimal free resolution of  $K$ . In particular,  $\mathrm{cx}_S K = 0$  if and only if  $K$  has finite projective dimension.

**Remark 4.** The first inequality below follows from a result of Gulliksen [5, (3.1)], cf. [1, (1.3)]; the other two inequalities are established in [1, (1.3)].

$$\begin{aligned}
 (4) \quad & \mathrm{cx}_S(K, L) \leq \mathrm{codim} S \\
 (5) \quad & \mathrm{cx}_S K + \mathrm{cx}_S L - \mathrm{codim} S \leq \mathrm{cx}_S(K, L) \leq \min\{\mathrm{cx}_S K, \mathrm{cx}_S L\}
 \end{aligned}$$

When  $\mathrm{Tor}_n^S(K, L) = 0$  for all  $n \geq 1$  the following equality is proved in [8, (2.1)].

$$(6) \quad \mathrm{cx}_S K + \mathrm{cx}_S L = \mathrm{cx}_S(K \otimes_S L)$$

When  $K$  is Cohen-Macaulay, [1, (5.6.2), (5.6.8), (5.6.10)] imply the next equality.

$$(7) \quad \mathrm{cx}_S(K, L) = \mathrm{cx}_S(L, \mathrm{Ext}_S^{\dim S - \dim K}(K, S))$$

We relate complexity to the growth of lengths of torsion modules.

**Proposition 5.** *If  $K$  is a module of finite length over a complete intersection  $S$ , then for each finitely generated  $S$ -module  $L$  there are polynomials  $g_{\pm}(t) \in \mathbb{Q}[t]$  with*

$$\ell_S(\mathrm{Tor}_n^S(K, L)) = \begin{cases} g_+(n) & \text{for all } n = 2s \gg 0; \\ g_-(n) & \text{for all } n = 2s + 1 \gg 0; \end{cases}$$

$$\max\{\deg g_+(t), \deg g_-(t)\} = \mathrm{cx}_S(K, L).$$

*Proof.* Let  $E$  be an injective envelope of the  $S$ -module  $k$ . The functor  $(-)^{\vee} = \mathrm{Hom}_S(-, E)$  of Matlis duality is exact, so for each  $n \geq 0$  we obtain isomorphisms

$$\mathrm{Tor}_n^S(K, L)^{\vee} \cong \mathrm{Ext}_S^n(L, K^{\vee})$$

of  $S$ -modules. Since Matlis duality preserves lengths, we see that

$$\ell_S(\mathrm{Tor}_n^S(K, L)) = \ell_S(\mathrm{Ext}_S^n(L, K^{\vee})).$$

By Gulliksen [5, (3.1)], the graded  $S$ -module  $\mathcal{E} = \mathrm{Ext}_S^{\bullet}(L, K^{\vee})$  has a structure of finitely generated graded module over a polynomial ring  $S[\chi]$  with indeterminates  $\chi = \chi_1, \dots, \chi_c$  of degree 2. Since the  $S$ -module  $K$  has finite length, it is annihilated by  $\mathfrak{n}^m$  for some  $m \geq 1$ , so  $\mathfrak{n}^m \mathcal{E} = 0$ . It follows that  $\mathcal{E}$  is also a finitely generated graded module over the polynomial ring  $(S/\mathfrak{n}^m)[\chi]$ . The Hilbert-Serre Theorem now provides polynomials  $g_{\pm}(t) \in \mathbb{Q}[t]$  such that

$$\ell_S(\mathrm{Ext}_S^n(L, K^{\vee})) = \begin{cases} g_+(n) & \text{for all } n = 2s \gg 0; \\ g_-(n) & \text{for all } n = 2s + 1 \gg 0; \end{cases}$$

$$\max\{\deg g_+(t), \deg g_-(t)\} = \dim_{S[\chi]} \mathcal{E}.$$

Since  $(\mathfrak{n}S[\chi])^m \mathcal{E} = \mathfrak{n}^m \mathcal{E} = 0$ , we have the first equality in the following sequence:

$$\dim_{S[\chi]} \mathcal{E} = \dim_{S[\chi]}(\mathcal{E}/\mathfrak{n}\mathcal{E}) = \mathrm{cx}_S(L, K^{\vee}) = \mathrm{cx}_S(K, L).$$

The second comes from dimension theory. The third is a consequence of formula (7) and the isomorphism  $K^{\vee} \cong \mathrm{Ext}_S^{\dim S}(K, S)$ , due to the finite length of  $K$ .  $\square$

**Remark 6.** Let  $R$  be a local complete intersection of characteristic  $p$ , with  $\widehat{R} = Q/(\mathbf{x})$  where  $Q = k[[t]]$  and  $\mathbf{x} = x_1, \dots, x_c$  is a  $Q$ -regular sequence, cf. Remark 2.

The local ring  $S = \widehat{R} \otimes_Q {}^{\varphi}Q = Q/(\mathbf{x}^{\varphi})$  is complete intersection with  $\mathrm{codim} S = c$ . Let  $\sigma: S \rightarrow \widehat{R}$  be the canonical surjection, and let  $\rho$  denote the composition

$$R \xrightarrow{\iota} \widehat{R} = \widehat{R} \otimes_Q Q \xrightarrow{\widehat{R} \otimes_Q {}^{\varphi}} \widehat{R} \otimes_Q {}^{\varphi}Q = S.$$

As  $\iota$  and  $\varphi: Q \rightarrow Q$  are local flat homomorphisms, so is  $\rho$ . It satisfies  $\varphi \iota = \sigma \rho$ , so for every  $R$ -module  $M$  and for the  $S$ -module  $K = M \otimes_R {}^{\rho}S$  the following hold.

$$(8) \quad \mathrm{Tor}_n^R(M, {}^{\varphi}R) \otimes_R {}^{\iota}\widehat{R} \cong \mathrm{Tor}_n^S(K, {}^{\sigma}\widehat{R})$$

$$(9) \quad \mathrm{cx}_R(M) = \mathrm{cx}_S(K)$$

**Remark 7.** In view of the inclusion  $(\mathbf{x}^q) \subseteq (\mathbf{t})(\mathbf{x})$ , a result of Tate [12, Theorem 6] provides the first equality in the following sequence:

$$\sum_{n=0}^{\infty} \ell_S(\mathrm{Ext}_S^n(\sigma\widehat{R}, k))t^n = \frac{(1+t)^c}{(1-t^2)^c} = \frac{1}{(1-t)^c} = \sum_{n=0}^{\infty} \binom{n+c-1}{c-1} t^n.$$

Thus, we can determine the complexity of  $\sigma\widehat{R}$  as a module over  $S$ :

$$(10) \quad \mathrm{cx}_S \sigma\widehat{R} = \mathrm{codim} S$$

*Proof of Theorem. Part II.* Let  $M$  be a finitely generated  $R$ -module.

Suppose  $\mathrm{Tor}_j^R(M, {}^\circ R) = 0$  for some  $j \geq 1$ . By Part I of the proof,  $\mathrm{Tor}_n^R(M, {}^\circ R) = 0$  for all  $n \geq j$ . Replacing  $M$  by a syzygy, we may assume  $j = 1$ . We then obtain

$$\mathrm{cx}_R(M) + \mathrm{codim} S = \mathrm{cx}_S(K) + \mathrm{cx}_S(\sigma\widehat{R}) = \mathrm{cx}_S(K \otimes_S \sigma\widehat{R}) \leq \mathrm{codim} S$$

using formulas (9) and (10), (8) and (6), (4); thus,  $\mathrm{cx}_R M = 0$ , that is,  $\mathrm{pd}_R M < \infty$ .

Suppose  $\ell_R M < \infty$ . Completion preserves length and projective dimension, so we may assume  $R = \widehat{R}$ . By [2, (8.1)] there exist polynomials  $b_{\pm}(t) \in \mathbb{Q}[t]$  with

$$\ell_R(\mathrm{Tor}_n^R(M, k)) = \begin{cases} b_+(n) & \text{for all } n = 2s \gg 0; \\ b_-(n) & \text{for all } n = 2s + 1 \gg 0; \end{cases}$$

$$\deg b_+(t) = \deg b_-(t) = \mathrm{cx}_R M.$$

Note next the equalities  $\mathrm{cx}_R M = \mathrm{cx}_S K = \mathrm{cx}_S(K, {}^\circ R)$ , due to formulas (9), (5), and (10). The desired properties of the limits in the theorem are now seen to follow from the expressions above and those in Proposition 5 with  $L = {}^\circ R$ .  $\square$

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