FINITE GENERATION OF HOCHSCHILD HOMOLOGY ALGEBRAS

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Abstract. We prove converses of the Hochschild-Kostant-Rosenberg Theorem, in particular: If a commutative algebra S is flat and essentially of finite type over a noetherian ring $\Bbbk$, and the Hochschild homology $\mathrm{HH}_*(S|\Bbbk)$ is a finitely generated S-*algebra* for shuffle products, then S is smooth over $\Bbbk$.

Introduction

Let S be a commutative algebra over a commutative noetherian ring $\Bbbk$.

Shuffle products on the Hochschild complex define the *Hochschild homology algebra* $\mathrm{HH}_*(S|\Bbbk)$, which is graded-commutative and is natural in S and $\Bbbk$, cf. [11], [23]. Since $\mathrm{HH}_0(S|\Bbbk)$ is S itself, and $\mathrm{HH}_1(S|\Bbbk)$ is the S-module of Kähler differentials $\Omega^1_{S|\Bbbk}$, there is a canonical homomorphism of graded algebras

$$\omega^*_{S|\Bbbk}\colon \textstyle\bigwedge^*_S \Omega^1_{S|\Bbbk} \to \mathrm{HH}_*(S|\Bbbk)$$

mapping differential forms to Hochschild homology. It provides a piece of the product: $\omega^n_{S|\Bbbk}$ is injective if $n!$ is invertible in S. Little more is known in general.

In a special case the story is complete. Recall that S is *regular* over $\Bbbk$ if it is flat, and the ring $S\otimes_{\Bbbk} k$ is regular for each homomorphism $\Bbbk\to k$ to a field k. The algebra S is *smooth* if it is regular and essentially of finite type, cf. [16].

If S is smooth over $\Bbbk$, then $\Omega^1_{S|\Bbbk}$ is projective and $\omega^*_{S|\Bbbk}$ is bijective.

This classical result is due to Hochschild, Kostant, and Rosenberg [19] when $\Bbbk$ is a perfect field, and can be extended, with some work, to cover noetherian rings. Using their homology theory of commutative algebras [1], [26], André and Quillen provide a generalization and a converse: a noetherian $\Bbbk$-algebra S is regular if and only if it is flat, the S-module $\Omega^1_{S|\Bbbk}$ is flat, and the map $\omega^*_{S|\Bbbk}$ is bijective, cf. [2].

Our main result explains why shuffle product structures have remained elusive. It establishes a conjecture of Vigué-Poirrier [32], proved by her and Dupont [32], [13] when S is positively graded and $S_0=\Bbbk$ is a field of characteristic zero.

Theorem on Finite Generation. *If S is a flat $\Bbbk$-algebra essentially of finite type and the S-algebra $\mathrm{HH}_*(S|\Bbbk)$ is finitely generated, then S is smooth over $\Bbbk$.*

As a consequence, S is smooth if $\omega^*_{S|\Bbbk}$ is surjective or, more generally, if the S-*module* $\mathrm{HH}_*(S|\Bbbk)$ is finite. The last result also follows from an earlier

Theorem on Semi-Rigidity. *If S is a flat $\Bbbk$-algebra essentially of finite type, and $\mathrm{HH}_{2i-1}(S|\Bbbk)=0=\mathrm{HH}_{2j}(S|\Bbbk)$ for some $i,j>0$, then S is smooth over $\Bbbk$.*

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When $\Bbbk$ is a field this is proved independently by Avramov and Vigué-Poirrier [8] in arbitrary characteristic, and by Campillo, Guccione, Guccione, Redondo, Solotar, and Villamayor [10] if $\operatorname{char}(\Bbbk)=0$. Rodicio [28] conjectured that Hochschild homology over a field $\Bbbk$ is *rigid*: If $\mathrm{HH}_m(S|\Bbbk)=0$ for some $m>0$, then $\mathrm{HH}_n(S|\Bbbk)=0$ for all $n>m$; he and Lago [20], [28] prove this when S is complete intersection, and Vigué-Poirrier [31] when S is positively graded and $\operatorname{char}(\Bbbk)=0$.

Over noetherian rings $\Bbbk$ semi-rigidity is established by Rodicio [29]. His crucial observation is that this property can be proved in the wider context of augmented commutative algebras, using a result of Avramov and Rahbar-Rochandel on *large homomorphisms* of local rings [22] to replace the specific constructions of resolutions over $S\otimes_{\Bbbk}S$, on which the approach in [8], [10] is based. On the other hand, Larsen and Lindenstrauss [21] show that $\mathrm{HH}_{2i-1}(S|\mathbb{Z})\neq 0=\mathrm{HH}_{2i}(S|\mathbb{Z})$ for any ring of algebraic integers $S\neq\mathbb{Z}$ and all $i>0$, so Hochschild homology over $\mathbb{Z}$ is *not rigid*.

In Sections 1 and 2 we use DG (=differential graded) homological algebra to study large homomorphisms, further developing results and ideas applied to Hochschild homology in [8], free resolutions in [7], [3], and André-Quillen homology in [6], [4]. As a bonus, we get a concise proof of a local semi-rigidity theorem.

Sections 3 and 4 are at the heart of our argument, and go a long way towards determining the structure of large homomorphisms with finitely generated Tor algebras. In positive residual characteristic the local semi-rigidity theorem easily yields the desired finiteness result. In characteristic zero, besides DG homological algebra we use the finiteness results on André-Quillen homology from [6], [4]; the architecture of our proof mirrors, to some extent, the topological approach in [13], viewed trough the looking glass [5] between local algebra and rational homotopy.

We return to Hochschild homology in the last two sections.

In Section 5 we put together our local results to prove the theorems above. We also show by various examples that their hypotheses cannot be significantly relaxed.

In Section 6 we study nilpotence properties of shuffle products in Hochschild homology. When $\Bbbk$ is a field of characteristic 0 and S a *locally complete intersection* $\Bbbk$-algebra essentially of finite type, we prove that Hochschild homology is *nilpotent*: There is an integer $s\geq 1$ such that $\mathrm{HH}_{\geqslant 1}(S|\Bbbk)^s=0$. We provide examples that illustrate that this need not hold when $\Bbbk$ is a field of positive characteristic. On the other hand, the presence of divided powers on Hochschild homology entails that it is *nil* for *any algebra of positive characteristic*: If $qS=0$, then $w^q=0$ for each $w\in\mathrm{HH}_{\geqslant 1}(S|\Bbbk)$. In an earlier version of this paper we had asked whether Hochschild homology is also nil when $\Bbbk$ is a field of characteristic 0, and suggested $S=\Bbbk[x,y]/(x^2,xy,y^2)$ as a test case; Löfwall and Sköldberg, and independently Larsen and Lindenstrauss, showed that if $\Bbbk$ is a field of characteristic 0 then $\mathrm{HH}_*(S|\Bbbk)$ is *not nil*. Thus, the general form of the Theorem on Finite Generation is not a corollary of the Theorem on Semi-Rigidity.

1. DG algebras

Let $(P,\mathfrak{p},k)$ be a local ring P with maximal ideal $\mathfrak{p}$ and residue field $k=P/\mathfrak{p}$.

In this paper DG algebras over P are assumed to be *graded commutative*: if a is an element of degree i and b is one of degree j, then $ab=(-1)^{ij}ba$, and $a^2=0$ when i is odd. A morphism that induces an isomorphism in homology is called a *quasiisomorphism*, and it is often marked by the appearance of the symbol $\simeq$ next to its arrow. Details on DG algebra can be found in [25], [5, §1], [3, §1].

Let A be a DG algebra over P. The underlying graded P-algebra is denoted $A^\natural$. A *semifree extension* $A[X]$ is a DG algebra whose differential extends that of A, and such that $A[X]^\natural$ is isomorphic to the tensor product of $A^\natural$ with the exterior algebra on a free P-module with basis $\bigsqcup_{i\geqslant 1} X_{2i-1}$ and the symmetric algebra on a free P-module with basis $\bigsqcup_{i\geqslant 1} X_{2i}$; we further assume that each X_n is finite.

A semifree extension $P[X]$ is *minimal* if its differential is *decomposable*:

$$\partial(X) \subseteq (\mathfrak{p} + (X))^2 .$$

In a more detailed notation, this condition may be restated as

$$\partial(X_1) \subseteq \mathfrak{p}^2 \qquad \text{and} \qquad \partial(X_{n+1}) \subseteq \mathfrak{p}X_n + \sum_{i=1}^{n-1} PX_iX_{n-i} \qquad \text{for} \quad n \geq 1 .$$

We also need a different type of algebra extension, where *divided powers* variables, rather than polynomial ones, are adjoined in even degrees. An algebra obtained by this procedure is called a *semifree Γ-extension* of A, denoted $A\langle X\rangle$, and we say that X is a set of *Γ-variables* over A; for details cf. [30], [17, §1.1], [3, §7].

The results of this section elaborate on several earlier ones. Part (2) of the next theorem extends [7, (1.10)] and [3, (7.2.9)], while part (3) generalizes [3, (6.3.4)].

1.1. Theorem. *Let $\phi\colon P[X] \to Q[Y]$ be a surjective morphism of semifree extensions of regular local rings $(P, \mathfrak{p}, k)$ and $(Q, \mathfrak{q}, k)$ such that $P[X]$ is minimal.*

1. *There exists a set of variables $\widetilde{Y} \sqcup Z$ over P, such that $P[X] = P[\widetilde{Y}, Z]$, ϕ maps $\widetilde{Y}$ bijectively to Y, and $\operatorname{Ker}\phi = (\boldsymbol{p}, Z)P[X]$, where $\boldsymbol{p}$ is a regular sequence in $\mathfrak{p}$ that is linearly independent modulo $\mathfrak{p}^2$.*
2. *The morphism ϕ can be factored as*

$$P[\widetilde{Y}, Z] \xhookrightarrow{\ \iota\ } P[\widetilde{Y}, Z]\langle U\rangle \xrightarrow{\ \widetilde{\phi}\ } Q[Y]$$

where $\widetilde{\phi}$ is a quasiisomorphism, ι is an adjunction of a set of Γ-variables

$$U = \{u_z \mid z \in \boldsymbol{p} \sqcup Z \text{ and } \deg(u_z) = \deg(z) + 1\} ,$$

and the differential of $P[\widetilde{Y}, Z]\langle U\rangle$ has the property that

$$\partial(u_z) = z \qquad \textit{for} \qquad z \in \boldsymbol{p} ;$$

$$\partial(u_z) - z \in (\mathfrak{p}, \widetilde{Y}_{<j}, Z_{<j})P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle \qquad \textit{for} \qquad z \in Z_j .$$

3. *The module of cycles of $P[X]\langle U\rangle = P[\widetilde{Y}, Z]\langle U\rangle$ satisfies*

$$\mathrm{Z}_{\geqslant 1}(P[X]\langle U\rangle) \subseteq (\mathfrak{p} + (X))P[X]\langle U\rangle .$$

The next result generalizes [3, (7.2.7)].

1.2. Theorem. *In addition to the hypotheses of the preceding theorem, assume that $Q = k$ and $\mathrm{H}_{\geqslant s}(P[X]) = 0$ for some positive integer s. In that case*

$$(\mathrm{H}_{\geqslant 1}(k[Y]))^{s+\dim P} = 0 .$$

In the proofs of the theorems, and in later arguments, we need a few lemmas.

An element $z \in A$ is said to be *regular* if $\deg(z)$ is even and it is a non-invertible non-zero-divisor, or if $\deg(z)$ is odd and $\operatorname{Ann}_A(z) = (z)$. By extension, a finite sequence $z_1, \dots, z_j$ in A is *regular* if z_i is regular in $A/(z_1, \dots, z_{i-1})$ for $1 \leq i \leq j$.

1.3. Lemma. *If Z is a regular sequence of cycles in a DG algebra A, then the canonical surjection $A\langle W \,|\, \partial(W) = Z\rangle \to A/(Z)$ is a quasiisomorphism.*

Proof. An obvious induction shows that we may restrict to $A\langle w \,|\, \partial(w) = z\rangle$. Filtering $A\langle w\rangle$ by the internal degree of A, we get a spectral sequence with

$$ {}^0\mathrm{E}_{p,q} = A^\natural\langle w \,|\, \partial(w) = z\rangle_{p,q} \implies \mathrm{H}_{p+q}(A\langle w\rangle) $$

and differential defined by ${}^0\mathrm{d}(A^\natural) = 0$ and ${}^0\mathrm{d}(w) = z$. The regularity of z implies ${}^1\mathrm{E}_{p,q} = 0$ if $p \neq 0$ and ${}^1\mathrm{E}_{0,q} = A/(z)$, hence $\mathrm{H}_n(A\langle w\rangle) = {}^2\mathrm{E}_{0,n} = \mathrm{H}_n(A/(z))$. □

Let C be a DG module over a DG algebra A. For $r \in \mathbb{Z}$ the r'th *shift* of C is the DG module $\Sigma^r C$ having $(\Sigma^r C)_n = C_{n-r}$ for all n, differential $\partial\big(\Sigma^r(c)\big) = \Sigma^r\big((-1)^r\partial(c)\big)$ and action $a\Sigma^r(c) = (-1)^i\Sigma^r(ac)$ for $a \in A_i$, where $\Sigma^r\colon C \to \Sigma^r C$ is the degree r map sending $c \in C_{n-r}$ to $c \in (\Sigma^r C)_n$.

We use this construction to show that the adjunction of a finite package of *exterior* variables preserves finiteness properties.

1.4. Lemma. *Let A be a DG algebra, Z a finite set of cycles of even degree, and*

$$ A\langle W\rangle = A\langle W \,|\, \partial(W) = Z\rangle \,. $$

1. *If $\mathrm{H}_n(A) = 0$ for $n \geq s$, then $\mathrm{H}_n(A\langle W\rangle) = 0$ for $n \geq s + \sum_{w\in W}\deg(w)$.*
2. *If the algebra $\mathrm{H}_*(A)$ is noetherian, then the graded $\mathrm{H}_*(A)$-module $\mathrm{H}_*(A\langle W\rangle)$ is finite and annihilated by $\mathrm{cls}(z)$ for all $z \in Z$.*

Proof. The inclusions of DG algebras $A \subset A\langle w_1\rangle \subset \cdots \subset A\langle W\rangle$ show that it suffices to treat the case $A\langle W\rangle = A\langle w \,|\, \partial(w) = z\rangle$. We then have an exact sequence

$$ 0 \to A \xrightarrow{\iota} A\langle w\rangle \xrightarrow{\theta} \Sigma^r A \to 0 $$

where ι is the inclusion and $\theta(a + wb) = \Sigma^r(b)$. The homology exact sequence

$$ \Sigma^{r-1}\,\mathrm{H}_*(A) \xrightarrow{\eth} \mathrm{H}_*(A) \xrightarrow{\mathrm{H}_*(\iota)} \mathrm{H}_*(A\langle w\rangle) \xrightarrow{\mathrm{H}_*(\theta)} \Sigma^r\,\mathrm{H}_*(A) $$

immediately implies (1). For (2) note that $\eth(\Sigma^{r-1}(h)) = \mathrm{cls}(z)h$ and $\mathrm{H}_*(\iota)$ is a homomorphism of algebras, hence $\mathrm{cls}(z)$ annihilates the graded $\mathrm{H}_*(A)$-module $\mathrm{H}_*(A\langle w\rangle)$. This module is finite because the $\mathrm{H}_*(A)$-modules $\mathrm{H}_*(A)$ and $\Sigma^r\,\mathrm{H}_*(A)$ are noetherian and the maps $\mathrm{H}_*(\iota)$ and $\mathrm{H}_*(\theta)$ are $\mathrm{H}_*(A)$-linear. □

Using [17, (1.3.5)] and induction, or referring to [3, (7.2.10)], we have

1.5. Lemma. *If $\phi\colon A \to B$ is a quasiisomorphism, and $Z \subseteq A$ is a set of cycles, then ϕ extends to a quasiisomorphism of DG algebras*

$$ \phi_W\colon A\langle W \,|\, \partial(W) = Z\rangle \to B\langle W \,|\, \partial(W) = \phi(Z)\rangle $$

such that $\phi_W(w) = w$ for each $w \in W$. □

1.6. Lemma. *Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be surjective homomorphisms of (graded) algebras, and set $I = \operatorname{Ker}\alpha$, $J = \operatorname{Ker}(\beta\alpha)$, $K = \operatorname{Ker}\beta$. If the induced map $\mathrm{Tor}_2^{\phi}(C,C)\colon \mathrm{Tor}_2^A(C,C) \to \mathrm{Tor}_2^B(C,C)$ is surjective, then the exact sequence $0 \to I \to J \to K \to 0$ induces an exact sequence of (graded) C-modules*

$$ 0 \to I/IJ \to J/J^2 \to K/K^2 \to 0\,. $$

Proof. The standard change of rings spectral sequence with

$$ {}^2\mathrm{E}_{p,q} = \mathrm{Tor}_p^B(\mathrm{Tor}_q^A(B,C),C) \implies \mathrm{Tor}_{p+q}^A(C,C) $$

yields an exact sequence of graded C-modules

$$\operatorname{Tor}_2^A(C,C) \xrightarrow{\operatorname{Tor}_2^{\alpha}(C,C)} \operatorname{Tor}_2^B(C,C) \xrightarrow{\eth_2}$$
$$\operatorname{Tor}_1^A(B,C) \xrightarrow{\operatorname{Tor}_1^A(\beta,C)} \operatorname{Tor}_1^A(C,C) \xrightarrow{\operatorname{Tor}_1^{\alpha}(C,C)} \operatorname{Tor}_1^B(C,C) \longrightarrow 0\,.$$

Since $\operatorname{Tor}_2^{\alpha}(C,C)$ is surjective, we have $\eth_2 = 0$. Canonical isomorphisms identify the tail of the exact sequence above with the desired exact sequence. □

We also need a special case of Theorem 1.1, proved in [3, (7.2.9)].

1.7. For each minimal semifree extension $P[X]$ of a regular local ring $(P,\mathfrak{p},k)$ the surjective homomorphism $P \to k$ can be factored as $P[X] \hookrightarrow P[X]\langle X'\rangle \xrightarrow{\simeq} k$, where

$$\operatorname{card}(X'_n) = \begin{cases} \dim P & \text{for } n = 1\,; \\ \operatorname{card}(X_{n-1}) & \text{for } n \geq 2\,; \end{cases}$$
$$\partial(P[X]\langle X'\rangle) \subseteq (\mathfrak{p} + (X))P[X]\langle X'\rangle\ .$$

Proof of Theorem 1.1. The argument is broken down into several steps.

Step 1. $P[X] = P[\widetilde{Y}, Z]$ where $\widetilde{Y} \sqcup Z$ is a set of variables over P, ϕ maps $\widetilde{Y}$ bijectively to Y, and $\operatorname{Ker}\phi = (\boldsymbol{p}, Z)P[X]$, where $\boldsymbol{p}$ is a regular sequence in $\mathfrak{p}$ that is linearly independent modulo $\mathfrak{p}^2$.

Since $\phi_0 \colon P \to Q$ is a surjective homomorphism of regular local rings, $\operatorname{Ker}\phi_0$ is minimally generated by a set $\boldsymbol{p}$ that is linearly independent modulo $\mathfrak{p}^2$. Thus, the morphism ϕ factors as a composition of surjective morphisms

$$P[X] \longrightarrow\!\!\!\!\!\rightarrow P[X]/(\boldsymbol{p}) = Q[X] \xrightarrow{\alpha}\!\!\!\!\!\rightarrow Q[Y]\,.$$

Since the graded Q-algebra $Q[Y]^\natural$ is free, the surjective homomorphism of graded Q-algebras $\alpha^\natural \colon Q[X]^\natural \to Q[Y]^\natural$ is split by a homomorphism of graded Q-algebras $\sigma \colon Q[Y]^\natural \to Q[X]^\natural$. It follows that $\operatorname{Tor}_*^{\alpha^\natural}(Q,Q) \circ \operatorname{Tor}_*^{\sigma}(Q,Q)$ is the identity map of $\operatorname{Tor}_*^{Q[Y]^\natural}(Q,Q)$, so in particular $\operatorname{Tor}_2^{\alpha^\natural}(Q,Q)$ is surjective. Lemma 1.6 applied to $\alpha^\natural$ and $\beta \colon Q[Y]^\natural \to Q$ produces an exact sequence of graded Q-modules

$$0 \to (I/(X)I)^\natural \to (QX)^\natural \to (QY)^\natural \to 0$$

where $I = \operatorname{Ker}\alpha$. For each $j \geq 1$, choose in PX_j a set $\widetilde{Y}_j$ that ϕ maps bijectively onto Y_j, and a set $Z_j \subseteq I$ whose image in $I/(X)I$ is a basis of that Q-module. Thus, $\widetilde{Y} \sqcup Z$ generates the ideal of elements of positive degree of the graded Q-algebra $Q[X]^\natural$, and hence is a generating set of the algebra. Nakayama's Lemma then implies that $\widetilde{Y} \sqcup Z$ generates the P-algebra $P[X]^\natural$. We conclude from the equalities $\operatorname{card}(\widetilde{Y}_j) + \operatorname{card}(Z_j) = \operatorname{card}(X_j)$ that $\widetilde{Y} \sqcup Z$ is a set of variables over P.

Step 2. ϕ factors as $P[\widetilde{Y}, Z] \xhookrightarrow{\iota} P[\widetilde{Y}, Z]\langle U\rangle \xrightarrow{\widetilde{\phi}} Q[Y]$, where

$$U_1 = \{u_z \,|\, z \in \boldsymbol{p}\} \quad \text{and} \quad \partial(u_z) = z \quad \text{for} \quad z \in \boldsymbol{p}\,;$$
$$U_{j+1} = \{u_z \,|\, z \in Z_j\} \quad \text{and} \quad \partial(u_z) - z \in P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle \quad \text{for} \quad z \in Z_j\,.$$

First, we factor ϕ as a composition of morphisms of DG algebras

$$P[\widetilde{Y}, Z] \xhookrightarrow{\iota^{(1)}} P[\widetilde{Y}, Z]\langle U_1 \,|\, \partial(U_1) = \boldsymbol{p}\rangle \xrightarrow[\simeq]{\pi^{(1)}}\!\!\!\!\!\rightarrow Q[\widetilde{Y}, Z] \xrightarrow{\varkappa^{(1)}}\!\!\!\!\!\rightarrow Q[Y]$$

where $\iota^{(1)}$ is an adjunction of a set U_1 of Γ-variables over $P[\widetilde{Y}, Z]$ with $\operatorname{card}(U_1) = \operatorname{card}(\boldsymbol{p})$, and $\pi^{(1)}$ is the canonical surjection with $\operatorname{Ker}\pi^{(1)} = (\boldsymbol{p}, U_1)$; since $\boldsymbol{p}$ is a regular sequence, $\pi^{(1)}$ is a quasiisomorphism by Lemma 1.3.

Assume by induction that for some $j \geq 1$ we have a factorization

$$P[\widetilde{Y}, Z] \xhookrightarrow{\iota^{(j)}} P[\widetilde{Y}, Z]\langle U_{\leqslant j}\rangle \xrightarrow[\simeq]{\pi^{(j)}} Q[\widetilde{Y}, Z_{\geqslant j}] \xrightarrow{\varkappa^{(j)}} Q[Y]$$

such that the following hold:

- $\iota^{(j)}$ is an adjunction of a set $U_{\leqslant j}$ of Γ-variables, extending U_1, over $P[\widetilde{Y}, Z]$;
- $U_{i+1} = \{u_z \mid z \in Z_i\}$ for $1 \leq i \leq j-1$;
- $\partial(u_z) - \iota^{(j)}(z) \in P[\widetilde{Y}_{<i}, Z_{<i}]\langle U_{\leqslant i}\rangle$ for each $z \in Z_i$ and $1 \leq i \leq j-1$;
- $\pi^{(j)}$ is a surjective quasiisomorphism with kernel generated by the sets $\boldsymbol{p}$, $Z_{<j}$, $U_{\leqslant j}$, and $\{u^{(q)} | u \in U_{2h},\ 2h \leq j,\ q \geq 2\}$;
- $\varkappa^{(j)}$ is the canonical surjection with kernel generated by the set $Z_{\geqslant j}$.

Since $\varkappa_i^{(j)}$ is bijective for $i < j$, the equalities $\varkappa_{j-1}^{(j)}\partial_j(Z_j) = \partial_j\varkappa_j^{(j)}(Z_j) = 0$ show that $Z_j \subseteq Q[\widetilde{Y}, Z_{\geqslant j}]$ consists of cycles. As $\pi^{(j)}$ is a surjective quasiisomorphism, for each $z \in QZ_j$ there is a cycle $\widehat{z} \in P[\widetilde{Y}, Z]\langle U_{\leqslant j}\rangle$ with $\pi^{(j)}(\widehat{z}) = z$. By the description of $\operatorname{Ker}\pi^{(j)}$, there are elements $a_y, b_{z'}$ in $\boldsymbol{p}$ such that

$$\widehat{z} = z + \sum_{y \in \widetilde{Y}_j} a_y y + \sum_{z' \in Z_j \smallsetminus z} b_{z'} z' + w \quad \text{with} \quad w \in P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle\ .$$

Since $\partial(U_1) = \boldsymbol{p}$, we can further find elements $u_y, v_{z'} \in PU_1$ such that $\partial(u_y) = a_y$ and $\partial(v_{z'}) = b_{z'}$. Therefore, the cycle $\widehat{z}$ is homologous to a cycle

$$\widetilde{z} = \widehat{z} - \sum_y \partial(u_y y) - \sum_{z'} \partial(v_{z'} z') = z + \sum_y u_y \partial(y) + \sum_{z'} v_l \partial(z') + w$$

that satisfies $\pi^{(j)}(\widetilde{z}) = z$ and $\widetilde{z} - z \in P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle$.

Setting $\widetilde{Z}_j = \{\widetilde{z} | z \in Z_j\}$ we form a commutative diagram

$$\begin{array}{ccc}
P[\widetilde{Y}, Z] & \xlongequal{\qquad\qquad} & P[\widetilde{Y}, Z] \\
{\scriptstyle \iota^{(j)}}\big\downarrow & & \big\downarrow{\scriptstyle \iota^{(j+1)}} \\
P[\widetilde{Y}, Z]\langle U_{\leqslant j}\rangle & \hookrightarrow & P[\widetilde{Y}, Z]\big\langle U_{\leqslant j+1} \,\big|\, \partial\big(U_{(j+1)}\big) = \widetilde{Z}_j\big\rangle \\
{\scriptstyle \pi^{(j)}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \pi^{(j)}_{U_{(j+1)}}} \\
Q[\widetilde{Y}, Z_{\geqslant j}] & \hookrightarrow & Q[\widetilde{Y}, Z_{\geqslant j}]\big\langle U_{(j+1)} \,\big|\, \partial\big(U_{(j+1)}\big) = Z_j\big\rangle \\
{\scriptstyle \varkappa^{(j)}}\big\downarrow & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \upsilon^{(j+1)}} \\
Q[Y] & \xleftarrow{\ \varkappa^{(j+1)}\ } & Q[\widetilde{Y}, Z_{\geqslant j+1}]
\end{array}$$

of morphisms of DG algebras; the quasiisomorphism $\upsilon^{(j+1)}$ is provided by 1.3 because Z_j, being part of a set of variables in $Q[\widetilde{Y}, Z_{\geqslant j}]$, is a regular sequence; the quasiisomorphism $\pi^{(j)}_{U_{(j+1)}}$ comes from Lemma 1.5. To finish the inductive construction, set $\pi^{(j+1)} = \upsilon^{(j+1)} \circ \pi^{(j)}_{U_{(j+1)}}$.

As $\pi = \varinjlim \pi^{(j)}$ stays a surjective quasiisomorphism and $\varkappa = \varinjlim \varkappa^{(j)}$ becomes an isomorphism, ϕ factors through the surjective quasiisomorphism $\widetilde{\phi} = \varkappa\pi$.

Step 3. $\mathrm{Z}_{\geqslant 1}(P[\widetilde{Y}, Z]\langle U\rangle) \subseteq (\mathfrak{p} + (\widetilde{Y}, Z))P[\widetilde{Y}, Z]\langle U\rangle$.

For this argument it is convenient to revert to the notation $P[X]$.

Since $P[X]$ is minimal and ϕ is surjective, the DG algebra $Q[Y]$ is minimal. Choose by 1.7 a quasiisomorphism $Q[Y]\langle Y'\rangle \to k$ and extend it by Lemma 1.5 to a quasiisomorphism $P[X]\langle U, Y'\rangle \to Q[Y]\langle Y'\rangle$. If $P[X]\langle X'\rangle \to k$ is a quasiisomorphism given by 1.7, then $P[X]\langle X'\rangle$ and $P[X]\langle U, Y'\rangle$ are quasiisomorphic DG modules over $P[X]$, cf. [3, (1.3.1)]. By [3, (1.3.3)] we then get a quasiisomorphism

$$k\langle X'\rangle = k \otimes_{P[X]} P[X]\langle X'\rangle \simeq k \otimes_{P[X]} P[X]\langle U, Y'\rangle = k\langle U, Y'\rangle \ .$$

As $\partial(k\langle X'\rangle) = 0$, we obtain (in)equalities of formal power series

$$\begin{aligned}
\prod_{i=1}^{\infty}(1-(-t)^i)^{(-1)^{i-1}\operatorname{card}(X_i')} &= \sum_n \operatorname{rank}_k k\langle X'\rangle_n t^n \\
&= \sum_n \operatorname{rank}_k \mathrm{H}_n(k\langle X'\rangle) t^n \\
&= \sum_n \operatorname{rank}_k \mathrm{H}_n(k\langle U, Y'\rangle) t^n \\
&\preccurlyeq \sum_n \operatorname{rank}_k k\langle U, Y'\rangle_n t^n \\
&= \prod_{i=1}^{\infty}(1-(-t)^i)^{(-1)^{i-1}(\operatorname{card}(U_i)+\operatorname{card}(Y_i'))}
\end{aligned}$$

Applying successively 1.7 for $P[X]$, Step 2, and 1.7 for $Q[Y]$ we get

$$\operatorname{card} X_i' = \begin{cases} \dim P = \operatorname{card} U_1 + \dim Q \quad = \operatorname{card} U_1 + \operatorname{card} Y_1' \text{for} & i = 1\,; \\ \operatorname{card} X_{i-1} = \operatorname{card} U_i + \operatorname{card} Y_{i-1} = \operatorname{card} U_i + \operatorname{card} Y_i' \text{ for} & i \geq 2\,. \end{cases}$$

Thus, $\operatorname{rank}_k \mathrm{H}_n(k\langle U, Y'\rangle) = \operatorname{rank}_k \mathrm{H}_n(k\langle U, Y'\rangle)$ for all n, so $\partial(k\langle U, Y'\rangle) = 0$. Put in other terms, we have $\partial(P[X]\langle U, Y'\rangle) \subseteq (\mathfrak{p} + (X))P[X]\langle U, Y'\rangle$. Since $P[X]\langle U\rangle$ is a DG subalgebra of $P[X]\langle U, Y'\rangle$ and the latter is acyclic, we have

$$\begin{aligned}
\mathrm{Z}_{\geqslant 1}(P[X]\langle U\rangle) &= \mathrm{Z}_{\geqslant 1}(P[X]\langle U, Y'\rangle) \cap (P[X]\langle U\rangle_{\geqslant 1}) \\
&= \partial(P[X]\langle U, Y'\rangle) \cap (P[X]\langle U\rangle_{\geqslant 1}) \\
&\subseteq (\mathfrak{p} + (X))P[X]\langle U, Y'\rangle \cap (P[X]\langle U\rangle) \\
&= (\mathfrak{p} + (X))P[X]\langle U\rangle
\end{aligned}$$

where the last equality arises from the freeness of $P[X]\langle U, Y'\rangle^{\natural}$ over $P[X]\langle U\rangle^{\natural}$.

Step 4. $\partial(u_z) - z \in (\mathfrak{p}, \widetilde{Y}_{<j}, Z_{<j})P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle$ for $z \in Z_j$.

Putting together the results of the last two steps, for $z \in Z_j$ we get

$$\begin{aligned}
\partial(u_z) - z &\in (\mathfrak{p} + (\widetilde{Y}, Z))P[\widetilde{Y}, Z]\langle U\rangle \cap P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle \\
&= (\mathfrak{p} + (\widetilde{Y}_{<j-1}, Z_{<j-1}))P[\widetilde{Y}_{<j}, Z_{<j}]\langle U_{\leqslant j}\rangle \ .
\end{aligned}$$

At this point, we have established all the assertions of the theorem. ∎

Proof of Theorem 1.2. Choose a minimal set $\boldsymbol{p}$ of generators of $\mathfrak{p}$. It contains $\dim P$ elements, so $\mathrm{H}_n(P[X]\langle W \,|\, \partial(W) = \boldsymbol{p}\rangle) = 0$ for $n \geq s + \dim P$ by Lemma 1.4. The morphism π factors through $P[X]\langle W\rangle \to P[X]\langle W\rangle/(W, \boldsymbol{p}) = k[X]$, the arrow is a quasiisomorphism by Lemma 1.3, and $k[X]$ is minimal. Thus, after changing notation we may assume that $P = k$, and (hence) $s + \dim P = s$.

Setting $J_n = 0$ for $n < s-1$, $J_{s-1} = \partial_s(k[X]_s)$, and $J_n = k[X]_n$ for $n \geq s$ we get a DG ideal J of $k[X]$, with $\mathrm{H}_*(J) = 0$. Let $k[X] \hookrightarrow k[X]\langle U\rangle \xrightarrow{\simeq} k[Y]$ be the factorization of ϕ given by Theorem 1.1. That theorem guarantees that the module of cycles $\mathrm{Z}_{\geqslant 1}(k[X]\langle U\rangle)$ is contained in $(X)k[X]\langle U\rangle$. As $k[X]\langle U\rangle^\natural$ is free over $k[X]^\natural$, it follows that $\mathrm{H}_*(Jk[X]\langle U\rangle) = 0$, and so we obtain

$$(\mathrm{Z}_{\geqslant 1}(k[X]\langle U\rangle))^s \subseteq \mathrm{Z}((X)^s k[X]\langle U\rangle) \subseteq \mathrm{Z}(Jk[X]\langle U\rangle) = \partial(Jk[X]\langle U\rangle)\,.$$

Since $k[X]\langle U\rangle \to k[Y]$ is a surjective quasiisomorphism, we have $\mathrm{Z}_{\geqslant 1}(k[Y]) = \pi(\mathrm{Z}_{\geqslant 1}(k[X]\langle U\rangle))$, and so $(\mathrm{H}_{\geqslant 1}(k[Y]))^s = 0$, as desired. □

2. Large homomorphisms

Following Levin [22], we say that a surjective homomorphism $\varphi\colon R \to S$ of local rings with residue field k is *large* if for each $n \in \mathbb{Z}$ it induces a surjective map

$$\mathrm{Tor}_n^{\varphi}(k,k)\colon\ \mathrm{Tor}_n^R(k,k) \to \mathrm{Tor}_n^S(k,k)\,.$$

For instance, if φ is *split* by a ring homomorphism $\psi\colon S \to R$ such that $\varphi\psi = \mathrm{id}_S$, then $\mathrm{Tor}_n^{\varphi}(k,k) \circ \mathrm{Tor}_n^{\psi}(k,k) = \mathrm{id}_{\mathrm{Tor}_n^S(k,k)}$ by functoriality, and hence φ is large.

The next result is the generalization [29] of the main theorems of [8], [10].

2.1. Theorem. *If $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},k)$ is a large homomorphism of local rings and $\mathrm{Tor}_n^R(S,S) = 0$ for some even positive n and some odd positive n, then $\mathrm{Ker}\,\varphi$ is generated by a regular sequence that extends to a minimal set of generators of $\mathfrak{m}$.*

The theorem is proved at the end of this section. The major ingredient is the following result, which plays a fundamental role in the next section as well.

2.2. Theorem. *Let $\rho\colon P \to R$ and $\varphi\colon R \to S$ be surjective homomorphisms of local rings such that $(P,\mathfrak{p},k)$ is regular, $\mathrm{Ker}\,\rho \subseteq \mathfrak{p}^2$, and φ is large.*

There exist a regular local ring $(Q,\mathfrak{q},k)$, a homomorphism $\sigma\colon Q \to S$ with $\mathrm{Ker}\,\sigma \subseteq \mathfrak{q}^2$, and a commutative diagram of morphisms of DG algebras

$$\begin{array}{ccccc}
P[X] & \hookrightarrow & P[X]\langle U\rangle & \xrightarrow[\simeq]{\widetilde{\phi}} & Q[Y] \\
{\scriptstyle \widetilde{\rho}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \widetilde{\pi}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \widetilde{\sigma}}\big\downarrow{\scriptstyle \simeq} \\
R & \hookrightarrow & R\langle U\rangle & \xrightarrow[\simeq]{\widetilde{\varphi}} & S
\end{array}$$

where $\widetilde{\rho}_0 = \rho$, $\widetilde{\sigma}_0 = \sigma$, $\widetilde{\varphi}_0 = \varphi$, labeled maps are surjective quasiisomorphisms, and the following hold

$$\partial(P[X]) \subseteq (\mathfrak{p} + (X))^2 P[X]\,; \qquad \partial(P[X]\langle U\rangle) \subseteq (\mathfrak{p} + (X))P[X]\langle U\rangle\,;$$
$$\partial(Q[Y]) \subseteq (\mathfrak{q} + (Y))^2 Q[Y]\,; \qquad \partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle\,.$$

Furthermore, the DG algebra $Q[Y]\langle U\rangle = Q[Y] \otimes_{P[X]} P[X]\langle U\rangle$ satisfies

$$\partial(U_1) = 0 \qquad \textit{and} \qquad \partial(U_{j+1}) \subseteq (\mathfrak{q} + (Y_{<j}))Q[Y_{<j}]\langle U_{\leqslant j}\rangle \quad \textit{for} \quad j \geq 1\,.$$

Before starting on the proof, we recall some properties of large homomorphisms.

2.3. Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},k)$ be a large homomorphism

2.3.1. Each minimal generating set of $\operatorname{Ker}\varphi$ is linearly independent modulo $\mathfrak{m}^2$.
Indeed, this follows from the exact sequence

$$0\to(\operatorname{Ker}\varphi)/\mathfrak{m}(\operatorname{Ker}\varphi)\to\mathfrak{m}/\mathfrak{m}^2\to\mathfrak{n}/\mathfrak{n}^2\to 0\,.$$

obtained by applying Lemma 1.6 to the ring homomorphisms $R\to S$ and $S\to k$.

2.3.2. The induced homomorphism of $\mathfrak{m}$-adic completions $\widehat{\varphi}\colon \widehat{R}\to\widehat{S}$ is large.
Indeed, the natural isomorphisms $\operatorname{Tor}^{\widehat{R}}_*(k,k)\cong\operatorname{Tor}^{R}_*(k,k)$ and $\operatorname{Tor}^{\widehat{S}}_*(k,k)\cong\operatorname{Tor}^{S}_*(k,k)$ imply that $\operatorname{Tor}^{\widehat{\varphi}}_*(k,k)$ is surjective whenever $\operatorname{Tor}^{\varphi}_*(k,k)$ is.

Proof of Theorem 2.2. Let $\boldsymbol{p}$ be a subset of $\mathfrak{p}$ that ρ maps bijectively onto a minimal generating set of $\operatorname{Ker}\varphi$. It follows from 2.3.1 that $\boldsymbol{p}$ is linearly independent modulo $\mathfrak{p}^2$, so the ring $(Q,\mathfrak{q},k)=P/(\boldsymbol{p})$ is regular. As $\boldsymbol{p}$ is in the kernel of $\varphi\rho$, this map induces a surjective homomorphism $\sigma\colon Q\to S$, with $\operatorname{Ker}\sigma\subseteq\mathfrak{q}^2$ by the choice of $\boldsymbol{p}$.

Next, factor ρ and σ as $P\hookrightarrow P[X]\xrightarrow{\widetilde{\rho}} R$ and $Q\hookrightarrow Q[Y]\xrightarrow{\widetilde{\sigma}} S$ with minimal DG algebras $P[X]$ and $Q[Y]$ and quasiisomorphisms $\widetilde{\rho}$ and $\widetilde{\sigma}$, cf. [3, (7.4.2)]. By [3, (2.1.9)] there exists a morphism of DG algebras $\phi\colon P[X]\to Q[Y]$ with $\mathrm{H}_*(\phi)=\varphi$.

Assuming for the moment that ϕ is surjective, use Theorem 1.1.1 to factor it as

$$P[X]\hookrightarrow P[X]\langle U\rangle\xrightarrow{\widetilde{\phi}} Q[Y]\,.$$

This is the top row of the desired diagram. The rest of the diagram represents base change along $\widetilde{\rho}$, using the identification $R\otimes_{P[X]}Q[Y]=S$. The maps $\widetilde{\rho}$, $\widetilde{\sigma}$, and $\widetilde{\phi}$ are quasiisomorphisms by construction. The map $\widetilde{\pi}$ has the same property because it is obtained by base change from a quasiisomorphism of DG modules whose underlying graded modules are free over $P[X]^\natural$, cf. [3, (1.3.2)]. The commutativity of the right hand square shows that $\widetilde{\varphi}$ is a quasiisomorphism. As $P[X]$ and $Q[Y]$ are minimal DG algebras, their differentials have the desired properties. By base change, we deduce from Theorem 1.1.2 that $\partial(R\langle U\rangle)\subseteq\mathfrak{m}R\langle U\rangle$, and that in $Q[Y]\langle U\rangle$ we have $\partial(U_1)=0$ and $\partial(U_{j+1})\subseteq(\mathfrak{q}+(Y_{<j}))Q[Y_{<j}]\langle U_{\leqslant j}\rangle$ for $j\geq 1$.

To finish the proof of the theorem, it remains to show that ϕ is surjective.

Let I and J be the augmentation ideals defined by the exact sequences

$$(*)\qquad 0\to I\to P[X]^\natural\to k\to 0\,;\qquad 0\to J\to Q[Y]^\natural\to k\to 0\,.$$

For each n we then get a diagram

$$\begin{array}{ccccccc}
\operatorname{Tor}^R_n(k,k) & \xleftarrow[\cong]{\operatorname{Tor}^{\widehat{\rho}}_n(k,k)} & \operatorname{Tor}^{P[X]}_n(k,k) & \xrightarrow{\alpha_n} & \operatorname{Tor}^{P[X]^\natural}_1(k,k)_{n-1} & \xrightarrow[\cong]{\gamma_n} & (I/I^2)_{n-1}\\
\Big\downarrow{\scriptstyle \operatorname{Tor}^{\varphi}_n(k,k)} & & \Big\downarrow{\scriptstyle \operatorname{Tor}^{\phi}_n(k,k)} & & \Big\downarrow{\scriptstyle \operatorname{Tor}^{\phi^\natural}_1(k,k)_{n-1}} & & \Big\downarrow{\scriptstyle \overline{\phi}_{n-1}}\\
\operatorname{Tor}^S_n(k,k) & \xleftarrow[\cong]{\operatorname{Tor}^{\widehat{\sigma}}_n(k,k)} & \operatorname{Tor}^{Q[Y]}_n(k,k) & \xrightarrow{\beta_n} & \operatorname{Tor}^{Q[Y]^\natural}_1(k,k)_{n-1} & \xrightarrow[\cong]{\delta_n} & (J/J^2)_{n-1}
\end{array}$$

where the Tor functors in the right hand square are those of Eilenberg and Moore [25], cf. also [5, §1], and the following hold:

- $\operatorname{Tor}^{\widehat{\rho}}_n(k,k)$ and $\operatorname{Tor}^{\widehat{\sigma}}_n(k,k)$ are bijective because $\widehat{\rho}$ and $\widehat{\sigma}$ are quasiisomorphisms.
- α_n and β_n are edge homomorphisms in the spectral sequences

$$(**)\qquad {}^1\mathrm{E}^A_{p,q}=\operatorname{Tor}^{A^\natural}_q(k,k)_p\implies\operatorname{Tor}^A_{p+q}(k,k)$$

for the DG algebras $A=P[X]$ and for $A=Q[Y]$, respectively.

- γ_n and δ_n are connecting maps in the exact sequences of Tor induced by $(*)$.
- The map $\operatorname{Tor}^{\varphi}_n(k,k)$ is surjective because φ is large.

Using a suitably bigraded version of 1.3, one readily sees that

$$\operatorname{Tor}^{P[X]^\natural}_p(k,k)_q \cong k\langle X''\rangle_{p,q}$$

where $X''_{p,q} = X'_{p+1}$ if $q = 1$ and $X''_{p,q} = \varnothing$ otherwise. On the other hand, we have

$$\operatorname{Tor}^{P[X]}_*(k,k) = \mathrm{H}_*(k \otimes_{P[X]} P[X]\langle X'\rangle) = k\langle X'\rangle$$

where the first equality holds by definition, and the second by 1.7. Thus, we get

$$\sum_{n=0}^{\infty}\Big(\sum_{p+q=n} \operatorname{rank}_k {}^1\mathrm{E}^{P[X]}_{p,q}\Big)t^n = \sum_{n=0}^{\infty}\big(\operatorname{rank}_k \operatorname{Tor}^{P[X]}_n(k,k)\big)t^n\,,$$

so the spectral sequence $(**)$ stops on the first page, and so α_n is surjective.

A similar argument establishes the surjectivity of β_n.

The diagram commutes because of the naturality of all the maps involved, so each $\overline{\phi}_n$ is surjective, and thus by Nakayama's Lemma ϕ is surjective, as desired. □

2.4. A DG algebra A over R is said to be a *DG Γ-algebra*, if each $a \in A$ of even positive degree has a sequence $\big(a^{(j)}\big)_{j\geqslant 1}$ of *divided powers* satisfying standard identities, cf. [17, (1.7.1), (1.8.1)], among them $a^{(0)} = 1$, $a^{(1)} = a$, as well as

$$a^{(i)}a^{(j)} = \frac{(i+j)!}{i!j!}a^{(i+j)} \quad\text{and}\quad \partial\big(a^{(j)}\big) = \partial(a)a^{(j-1)} \quad\text{for all}\quad i,j \geq 1\,.$$

Any semifree Γ-extension $R\langle U\rangle$ is a DG Γ-algebra in which the divided powers of the elements of U are the natural ones, cf. e.g. [17, (1.8.4)].

2.5. Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},k)$ be a surjective homomorphism of local rings.

A factorization of φ in the form $R \hookrightarrow R\langle U\rangle \xrightarrow{\simeq} S$ is called an *acyclic closure* of φ if $\partial(U_1)$ minimally generates $\operatorname{Ker}\varphi$, and $\{\operatorname{cls}(\partial(u))\,|\,u \in U_{n+1}\}$ is a minimal generating set of $\mathrm{H}_n(R\langle U_{\leqslant n}\rangle)$ for each $n \geq 1$. By [17, (1.9.5)], acyclic closures are unique up to isomorphism as DG Γ-algebras.

Thus, there is a "smallest" resolutions of S with a structure of semifree Γ-extension of R, and in that class it is "as unique as" a *minimal resolution* is among free resolutions. Here is a simple relation between the two concepts.

2.6. If $\mathrm{H}_*(R\langle U\rangle) \cong S$ and $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$, then $R\langle U\rangle$ is an acyclic closure of the homomorphism φ.

Indeed, if that fails, then for some $n \geq 0$ we have $\sum_{u\in U_{n+1}} r_u\partial(u) = \partial(v)$ with $r_u \in R$, not all $r_u \in \mathfrak{m}$, and $v \in R\langle U_{\leqslant n}\rangle$. It follows that $z = \sum_{u\in U_{n+1}} r_u u - v$ is a cycle in $\mathrm{Z}_{n+1}(R\langle U\rangle)$. Since $\mathrm{H}_{\geqslant 1}(R\langle U\rangle) = 0$, there exists an element $w \in R\langle U\rangle$ such that $\partial(w) = z \notin \mathfrak{m}R\langle U\rangle$, contradicting the minimality of $R\langle U\rangle$.

The converse of the last remark does not hold in general. One case when it does is for $S = R/\mathfrak{m}$, by a well known theorem of Gulliksen and Schoeller, cf. [17, (1.6.4)] or [3, (6.3.5)]. The homomorphism $R \to k$ is obviously large, so the next result constitutes a substantial extension. The proof here differs from those originally given, independently, by Avramov and by Rahbar-Rochandel, cf. [22, (2.5)].

2.7. Corollary. *If $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},k)$ is a large homomorphism, and $R\langle U\rangle$ is an acyclic closure of φ, then $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$.*

Proof. If R is complete, then by Cohen's Structure Theorem there is a surjective homomorphism $\rho\colon P \to R$, where $(P, \mathfrak{p}, k)$ is a regular local ring, and $\operatorname{Ker}\rho \subseteq \mathfrak{p}^2$. Theorem 2.2 now yields a DG algebra $R\langle U\rangle$ with $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$. By 2.6, it is an acyclic closure of φ, hence each acyclic closure has the desired property by 2.5.

In general, $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ is a large homomorphism by 2.3.2. If $R\langle U\rangle$ is an acyclic closure φ, then it is easy to see that $\widehat{R}\langle U\rangle = \widehat{R} \otimes_R R\langle U\rangle$ is one of $\widehat{\varphi}$, hence

$$\partial(R\langle U\rangle) \subseteq (R\langle U\rangle) \cap \partial(\widehat{R}\langle U\rangle) \subseteq (R\langle U\rangle) \cap \mathfrak{m}(\widehat{R}\langle U\rangle) = \mathfrak{m}R\langle U\rangle$$

where the second inclusion holds by the already established case. □

The non-vanishing homology classes below are also used in [8], [10], [29].

2.8. Corollary. *If $x_1, \dots, x_e$ minimally generate $\operatorname{Ker}\varphi$ and the Koszul complex $K = R\langle t_1, \dots, t_e \,|\, \partial(t_i) = x_i\rangle$ has $\mathrm{H}_1(K)$ minimally generated by c elements, then*

$$k\langle t_1, \dots, t_e\rangle \oplus k\langle u_1, \dots, u_c\rangle\, t_1 \cdots t_e \subseteq \operatorname{Tor}^R_*(S,S) \otimes_S k$$

where $\deg(t_i) = 1$ *for* $1 \le i \le e$ *and* $\deg(u_j) = 2$ *for* $1 \le j \le c$.

Proof. By 2.5, φ has an acyclic closure $R\langle U\rangle$ such that $U_1 = \{t_1, \dots, t_e\}$, $U_2 = \{u_1, \dots, u_c\}$, and $\operatorname{cls}(\partial(u_1)), \dots, \operatorname{cls}(\partial(u_2))$ minimally generate $\mathrm{H}_1(K)$. In the DG algebra $S\langle U\rangle = S \otimes_R R\langle U\rangle$ we have $\partial(U_1) = 0$ and $\partial(U_2) \subseteq SU_1$, hence

$$Z = S\langle t_1, \dots, t_e\rangle \oplus S\langle u_1, \dots, u_c\rangle\, t_1 \cdots t_e \subseteq S\langle U\rangle$$

is a submodule of cycles. By Corollary 2.7, $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$, so the composition

$$Z \otimes_S k \to \mathrm{H}_*(S\langle U\rangle) \otimes_S k \to \mathrm{H}_*(S\langle U\rangle \otimes_S k) = k\langle U\rangle$$

is injective. As $\mathrm{H}_*(S\langle U\rangle) = \operatorname{Tor}^R_*(S,S)$, this proves our assertion. □

Proof of Theorem 2.1. By hypothesis, $\varphi\colon R \to S$ is a large homomorphism with $\operatorname{Tor}^R_n(S,S) = 0$ for some even positive n and some odd positive n. By the preceding corollary we then have $c = 0$, that is, $\mathrm{H}_1(K) = 0$. This implies that the sequence $x_1, \dots, x_e$ is regular; it is linearly independent modulo $\mathfrak{m}^2$ by 2.3.1. □

3. Finite generation

Let $S \leftarrow R \to S'$ be homomorphisms of commutative rings. The $\pitchfork$-product of Cartan and Eilenberg [11, §XI.4] provides $\operatorname{Tor}^R_*(S,S')$ with a natural structure of graded-commutative algebra, which in degree 0 is the standard product on $S \otimes_R S'$. The product may be computed from any flat resolution of S over R. In particular, if A is a DG algebra with A_n a flat R-module for each n, $\mathrm{H}_0(A) \cong S$, and $\mathrm{H}_n(A) = 0$ for $n \ne 0$, then $\operatorname{Tor}^R_*(S,S') \cong \mathrm{H}_*(A \otimes_R S')$ as graded algebras.

In this section we focus on large homomorphisms of local rings with finitely generated Tor algebras. In non-zero characteristic we describe them completely.

3.1. Theorem. *Let $\varphi\colon R \to S$ be a surjective homomorphism of local rings.*
If R has residual characteristic $p > 0$, then the following are equivalent.

(i) *The S-algebra $\operatorname{Tor}^R_*(S,S)$ is finitely generated, and φ is large.*
(ii) *Each S-algebra S' defines a natural isomorphism of graded S'-algebras*

$$\operatorname{Tor}^R_*(S,S') \cong \textstyle\bigwedge_{S'}(\Sigma\, S'^{e})$$

with $e = \operatorname{edim} R - \operatorname{edim} S$, and φ is large.
(iii) *The ideal $\operatorname{Ker}\varphi$ is generated by an R-regular sequence that extends to a minimal system of generators of the maximal ideal of R.*

In characteristic zero we obtain only a partial description.

Recall that a local ring $(R, \mathfrak{m}, k)$ is a *local complete intersection* if in some (or, equivalently, any) Cohen presentation of its $\mathfrak{m}$-adic completion $\widehat{R}$ as $\widehat{R} \cong P/\mathfrak{a}$ with a regular local ring P, the ideal $\mathfrak{a}$ is generated by a P-regular sequence.

3.2. Theorem. *Let $\varphi\colon R \to S$ be a large homomorphism of local rings.*

If R has residual characteristic 0 and the S-algebra $\operatorname{Tor}_*^R(S,S)$ is finitely generated, then S has a minimal free resolution $R[U]$ with a finite set of variables U.

If furthermore R or S is a local complete intersection, then $U = U_1 \sqcup U_2$.

A conjecture of Quillen on the cotangent homology functors $\mathrm{D}_*(S\,|\,R;-)$ of André [1] and Quillen [26] predicts that the last assertion holds for all R and S.

3.3. Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, k)$ be a surjective homomorphism of local rings, let $R[U]$ be a semifree extension with $\mathrm{H}_*(R[U]) \cong S$, and let $\operatorname{char}(k) = 0$.

3.3.1. If A is a DG algebra over S, then $A[Y] \cong A\langle Y\rangle$ as DG algebras by a map that is the identity on A and on Y. Thus, in characteristic 0 we may replace Γ-extensions by free extensions. This is not only a matter of convenience: at a crucial step at the end of the proof of Theorem 3.2 we need to treat uniformly variables that were of different type at the moment of their adjunction.

3.3.2. By Quillen [26, (9.5)], for $\mathbf{L} = R[U]/(R + (U)^2 R[U])$ and each S-module N

$$\mathrm{D}_n(S\,|\,R;N) \cong \mathrm{H}_n(\mathbf{L} \otimes_R N) \quad \text{for} \quad n \in \mathbb{Z}\,.$$

In particular, if $R[U]$ is an acyclic closure of φ, cf. 2.5, then $\partial(\mathbf{L}) \subseteq \mathfrak{m}\mathbf{L}$, hence $\mathrm{D}_n(S\,|\,R;k) \cong kU_n$, so $\mathrm{D}_n(S\,|\,R;-) = 0$ for $n > m$ if and only if $U_n = \varnothing$ for $n > m$.

3.3.3. Assume that $\mathrm{D}_n(S\,|\,R;-) = 0$ for some integer m and all $n > m$.

Quillen [26, (5.6)] conjectures that $m \leq 2$, and the following is known:

(1) The conjecture holds if R or S is a local complete intersection by [4, (4.6)].

(2) If $\operatorname{fd}_R S < \infty$, that is, if S has a finite resolution by flat R-modules, then $\operatorname{Ker}\varphi$ is generated by a regular sequence by [6, Theorem A] or [4, (4.4)].

In view of the preceding remarks, we can reinterpret Theorem 3.2 as follows.

3.4. Theorem. *If $\varphi\colon R \to S$ is a large homomorphism of local rings of residual characteristic 0, and the S-algebra $\operatorname{Tor}_*^R(S,S)$ is finitely generated, then there exists an integer m such that $\mathrm{D}_n(S\,|\,R;-) = 0$ for $n > m$.*

If furthermore R or S is a local complete intersection, then $m \leq 2$. □

In positive characteristic the finiteness theorem 3.1 follows easily from the vanishing theorem 2.1, due to the existence of non-trivial operations on Tor.

3.5. If $S \leftarrow R \to S'$ are homomorphisms of commutative rings, then $\operatorname{Tor}_*^R(S,S')$ has a natural in all three arguments structure of Γ-algebra, in the sense of 2.4.

More precisely, let $R\langle U\rangle \to S$ be a quasiisomorphism. If z is a cycle in $S'\langle U\rangle = R\langle U\rangle \otimes S'$, then so is $z^{(n)}$, and its class in $\mathrm{H}_*(R\langle U\rangle \otimes_S S')$ depends only on $\operatorname{cls}(z)$, so $(\operatorname{cls}(z))^{(n)} = (\operatorname{cls}(z^{(n)}))$ yields a Γ-structure on $\operatorname{Tor}_*^R(S,S') \cong \mathrm{H}_*(R\langle U\rangle \otimes_S S')$; this structure does not depend on the choice of $R\langle U\rangle$, cf. [5, §1]. Thus, if $h \in \operatorname{Tor}_n^R(S,S')$ then $h^2 = 0$ for odd n and $h^q = q!h^{(q)}$ for even $n > 0$.

Proof of Theorem 3.1. In this proof $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, k)$ is a surjective homomorphism of local rings, and $\operatorname{char}(k) = p > 0$.

(ii) $\implies$ (i) is clear.

(i) $\implies$ (iii). Under our hypothesis, the algebra $\operatorname{Tor}^R_*(S,S)/p\operatorname{Tor}^R_*(S,S)$ is generated over S by finitely many elements of positive degree. By 3.5 their p'th powers are equal to 0, so $\operatorname{Tor}^R_n(S,S)=p\operatorname{Tor}^R_n(S,S)$ for $n\gg 0$, and hence $\operatorname{Tor}^R_n(S,S)=0$ by Nakayama's Lemma. Theorem 2.1 yields the desired conclusion.

(iii) $\implies$ (ii) is well known, but we include an argument for completeness. By hypothesis, $\operatorname{Ker}\varphi$ is minimally generated by an R-regular sequence $\boldsymbol{x}$ that is linearly independent modulo $\mathfrak{m}^2$. It follows that $\boldsymbol{x}$ has length $e=\operatorname{edim}R-\operatorname{edim}S$, and the Koszul complex $R\langle T\,|\,\partial(T)=\boldsymbol{x}\rangle$ yields

$$\operatorname{Tor}^R_*(S,S')=\mathrm{H}_*(R\langle T\rangle\otimes_R S')=S'\langle T\rangle=\textstyle\bigwedge_{S'}\Sigma S'^{e}\,.$$

Furthermore, $R\to k$ has an acyclic closure of the form $R\langle T,V\rangle$. By Lemma 1.3 the morphism $R\langle T,V\rangle\to R\langle T,V\rangle/(T,\boldsymbol{x})=S\langle V\rangle$ is a quasiisomorphism. As $R\langle T,V\rangle$ is a minimal resolution of k by the theorem of Gulliksen and Schoeller, recalled before Corollary 2.7, we see that $S\langle V\rangle$ is a minimal resolution of k over S, hence $\operatorname{Tor}^{\varphi}_*(k,k)\colon R\langle T,V\rangle\otimes_R k\to S\langle V\rangle\otimes_S k$ is surjective, that is, φ is large. □

Proof of Theorem 3.2. In this proof φ is a large homomorphism, as in 3.3, and the S-algebra $\operatorname{Tor}^R_*(S,S)$ is finitely generated.

The homomorphism $\widehat{\varphi}\colon\widehat{R}\to\widehat{S}$ is large by 2.3.2. The $\widehat{S}$-algebra $\operatorname{Tor}^{\widehat{R}}_*(\widehat{S},\widehat{S})$ is isomorphic to $\operatorname{Tor}^R_*(S,S)\otimes_S\widehat{S}$, and so finitely generated. Thus, we may assume that R is $\mathfrak{m}$-adically complete. Let $\rho\colon P\to R$ be a Cohen presentation with $(P,\mathfrak{p},k)$ regular and $\operatorname{Ker}\rho\subseteq\mathfrak{p}^2$. Theorem 2.2 now applies and we adopt its notation, modified in accordance with 3.3.1.

Since $P[X,U]^\natural$ is free over $P[X]^\natural$, the quasiisomorphism $Q[Y]\xrightarrow{\simeq}S$ yields

$$Q[Y,U]=Q[Y]\otimes_{P[X]}P[X,U]\xrightarrow{\simeq}S\otimes_{P[X]}P[X,U]=S\otimes_R R[U]\,.$$

By Theorem 2.2, the DG algebra $R[U]$ is a free resolution of S over R, so

$$\mathrm{H}_*(S\otimes_R R[U])=\operatorname{Tor}^R_*(S,S)\,.$$

We conclude that $\mathrm{H}_*(Q[Y,U])$ is finitely generated over S, say by the classes of $z_1,\dots,z_g$. Let $Z=\{z_1^2,\dots,z_g^2\}$, pick a minimal generating set $\boldsymbol{q}$ of $\mathfrak{q}$, and set

$$A=Q[Y,U,V,W\,|\,\partial(V)=Z\,;\partial(W)=\boldsymbol{q}]\,.$$

By Lemma 1.4, $\mathrm{H}_*(A)$ is a finite module over the noetherian ring $\mathrm{H}_*(Q[Y,U])$, and is annihilated by the ideal generated by $\boldsymbol{q}$ and $\{\operatorname{cls}(z_1)^2,\dots,\operatorname{cls}(z_g)^2\}$. This ideal has finite colength, so there is an integer s such that $\mathrm{H}_n(A)=0$ for all $n\geq s$.

Since Q is a regular local ring, the morphism $Q[W\,|\,\partial(W)=\boldsymbol{q}]\to k$ is a quasiisomorphism. As $Q[Y,U,V]$ is a bounded below complex of free Q-modules, it induces a quasiisomorphism of DG algebras

$$A=Q[Y,U,V,W]=Q[Y,U,V]\otimes_Q Q[W]\xrightarrow{\simeq}Q[Y,U,V]\otimes_Q k=k[Y,U,V]$$

so, in particular, $\mathrm{H}_n(k[Y,U,V])=0$ for all $n\geq s$. On the other hand,

$$\begin{aligned}\partial(Y)&\subseteq(Y)^2k[Y,U,V]\,;\\ \partial(U)&\subseteq(Y)(Y,U)k[Y,U,V]\,,\\ \partial(V)&\subseteq(Y,U)^2k[Y,U,V]\,,\end{aligned}$$

where the first two relations are provided by Theorem 2.2, and the last one holds by construction. Thus, $k[Y,U,V]$ is a minimal semifree extension of the field k and $\mathrm{H}_n(k[Y,U,V]) = 0$ for $n \geq s$. By Theorem 1.2, the surjective morphism

$$k[Y,U,V] \twoheadrightarrow k[Y,U,V]/(Y) = k[U,V]$$

shows that in $\mathrm{H}_*(k[U,V])$ the product of any s elements is equal to zero.

Setting $r = \max\{\deg(v) \,|\, v \in V\}$, we get an isomorphism of DG algebras

$$k[U,V] \cong k[U_{<r}, V] \otimes_k k[U_{\geqslant r}]$$

where $\partial(U) = 0$. In homology it induces an isomorphism of k-algebras

$$\mathrm{H}_*(k[U,V]) \cong \mathrm{H}_*(k[U_{<r}, V]) \otimes_k k[U_{\geqslant r}]\,.$$

We conclude that $\sum_{n=r}^{\infty} \operatorname{card} U_n < s$, hence U is finite, as desired.

If R or S is a complete intersection, then 3.3.3.2 yields $U_n = \varnothing$ for $n \neq 1, 2$. □

4. Split homomorphisms

The main result here is a structure theorem for certain split homomorphisms.

4.1. Theorem. *Let $S \xrightarrow{\psi} R \xrightarrow{\varphi} S$ be homomorphisms of local rings with $\varphi\psi = \mathrm{id}_S$. When R has residual characteristic 0 and $\operatorname{fd}_S R < \infty$ the following are equivalent.*

(i) *The S-algebra $\operatorname{Tor}_*^R(S,S)$ is finitely generated.*

(ii) *Each S-algebra S' defines a natural isomorphism of graded S'-algebras*

$$\operatorname{Tor}_*^R(S,S') \cong \textstyle\bigwedge_{S'}(\Sigma\, S'^{e}) \otimes_{S'} \operatorname{Sym}_{S'}(\Sigma^2 S'^{c})$$

with $e = \operatorname{edim} R - \operatorname{edim} S$ and $c = e - (\dim R - \dim S)$.

(iii) *The $(\operatorname{Ker}\varphi)$-adic completion of the ring R is isomorphic as an S-algebra to $S[[x_1,\dots,x_e]]/(\boldsymbol{f})$, where $\boldsymbol{f}$ is a length c regular sequence in $(x_1,\dots,x_e)^2$.*

The proof shows that the finiteness of the flat dimension $\operatorname{fd}_S R$ could be dropped if Quillen's conjecture 3.3.3 holds in characteristic 0. The arguments use *Tate complexes*, whose construction we recall next.

4.2. Let $\boldsymbol{x} = x_1,\dots,x_e$ and $\boldsymbol{f} = f_1,\dots,f_c$ be regular sequences in a commutative ring P that satisfy $f_j = \sum_{i=1}^{e} g_{ij}x_i$ for $j = 1,\dots,c$, set $R = P/(\boldsymbol{f})$ and $S = P/(\boldsymbol{x})$, and let $\pi\colon P \to R$ and $\varphi\colon R \to S$ be the canonical projections.

In the Koszul complex $R\langle T\rangle = R\langle t_1,\dots,t_e \,|\, \partial(t_i) = \pi(x_i)\rangle$ the elements $z_j = \sum_{i=1}^{e} \pi(g_{ij})t_i$ satisfy $\partial(z_j) = \sum_{i=1}^{e} \pi(g_{ij}x_i) = \pi(f_j) = 0$. Tate [30, Theorem 5], cf. also [17, (1.5.4)] or [3, (6.1.9)], proves that the DG algebra

$$R\langle T,U\rangle = R\langle T,U \,|\, \partial(t_i) = \pi(x_i)\,; \partial(u_j) = z_j\rangle$$

is a resolution of S over R. Thus, $\operatorname{Tor}_*^R(S,S) = \mathrm{H}_*(A)$ for the DG algebra

$$A = S \otimes_R R\langle T,U\rangle = S\big\langle T,U \,|\, \partial(T) = 0\,; \partial(u_j) = \sum_{i=1}^{e} a_{ij}t_i \ \text{ for } 1 \leq j \leq c\big\rangle$$

with $a_{ij} = \varphi\pi(g_{ij})$. When S contains a field of characteristic 0 the discussion works equally well with $R[T,U]$ in place of $R\langle T,U\rangle$, as noted in 3.3.1.

The preceding construction has strong implications for homology.

4.3. Proposition. *If R, S, A are as in 4.2 then the S-algebra $B = \operatorname{Tor}_*^R(S,S)$ has a bigrading with $B_n = \operatorname{Tor}_n^R(S,S) = \bigoplus_{\ell=0}^{n} B_n^{(\ell)}$, such that the following hold.*

(1) $B_n^{(\ell)} B_{n'}^{(\ell')} \subseteq B_{n+n'}^{(\ell+\ell')}$ *for all* ℓ, ℓ', n, n'.
(2) $B_0^{(0)} = \operatorname{Ker} \partial_0^{(0)} = A_0^{(0)} = S$ *and* $B_n^{(\ell)} = 0$ *unless* $0 \le 2\ell - n \le e$.
(3) $B_{2n}^{(n)} = \operatorname{Ker} \partial_{2n}^{(n)} \subseteq (0 : \mathfrak{j}) A_{2n}^{(n)}$ *for* $n \ge 1$, *where* $\mathfrak{j}$ *is the ideal in* S *generated by the* $c \times c$ *minors of the* $e \times c$ *matrix* (a_{ij}).

Proof. The products $t_{i_1} \cdots t_{i_r} \cdot u^{(\boldsymbol{s})}$, with $i_1 < \cdots < i_r$ and $u^{(\boldsymbol{s})} = u_1^{(s_1)} \cdots u_c^{(s_c)}$ for $\boldsymbol{s} = (s_1, \dots, s_c) \in \mathbb{N}^c$, form a basis of the graded S-module A. Assigning to such a product *upper degree* $\ell = r + s_1 + \cdots + s_c$, we turn A into a bigraded DG algebra $A = \bigoplus_{0 \leqslant \ell \leqslant n} A_n^{(\ell)}$ with $\partial(A_n^{(\ell)}) \subseteq A_{n-1}^{(\ell)}$, and $B = \mathrm{H}_*(A)$ inherits the bigrading.

(1) holds because $A_n^{(\ell)} A_{n'}^{(\ell')} \subseteq A_{n+n'}^{(\ell+\ell')}$.

(2) results from the equalities $A_0^{(0)} = S$ and $A_n^{(\ell)} = 0$ for $n < 2\ell - e$ or $n > 2\ell$.

(3) The relations $B_{2n}^{(n)} = \operatorname{Ker} \partial_{2n}^{(n)} \subseteq A_{2n}^{(n)}$ come from the equalities in (2).

The inclusion $\operatorname{Ker} \partial_2^{(1)} \subseteq (0 : \mathfrak{j}) U$ follows from Cramer's rule.

For $n \ge 2$ we use the basis $\{u^{(\boldsymbol{s})} : |\boldsymbol{s}| = n\}$ of $A_{2n}^{(n)}$ over S, where $|\boldsymbol{s}|$ stands for $s_1 + \cdots + s_c$. We denote $\boldsymbol{e}_j$ the j'th unit vector in $\mathbb{N}^c$, and make the convention that $u^{(\boldsymbol{s} - \boldsymbol{e}_j)} = 0$ if $s_j = 0$. In this notation, we have

$$\partial\Big(\sum_{|\boldsymbol{s}|=n} b_{\boldsymbol{s}} u^{(\boldsymbol{s})} \Big) = \sum_{|\boldsymbol{s}|=n} \sum_{j=1}^{c} b_{\boldsymbol{s}} \partial(u_j) u^{(\boldsymbol{s}-\boldsymbol{e}_j)} = \sum_{|\boldsymbol{s}|=n} \sum_{j=1}^{c} \sum_{i=1}^{e} b_{\boldsymbol{s}} a_{ij} t_i u^{(\boldsymbol{s}-\boldsymbol{e}_j)} .$$

For a cycle $\sum_{|\boldsymbol{s}|=n} b_{\boldsymbol{s}} u^{(\boldsymbol{s})}$ fix an index $\boldsymbol{s}$, choose h such that $s_h \neq 0$, and note that for each i the coefficient of $t_i u^{(\boldsymbol{s}-\boldsymbol{e}_h)}$ in the triple sum is equal to $\sum_{j=1}^{c} b_{\boldsymbol{s}-\boldsymbol{e}_h+\boldsymbol{e}_j} a_{ij}$. Since $\{t_i u^{(\boldsymbol{s}')} : |\boldsymbol{s}'| = n-1;\, i = 1, \dots, e\}$ is a basis of $A_{2n-1}^{(n)}$ over S, we see that

$$\partial_2^{(1)} \Big(\sum_{j=1}^{c} b_{\boldsymbol{s}-\boldsymbol{e}_h+\boldsymbol{e}_j} u_j \Big) = \sum_{j=1}^{c} b_{\boldsymbol{s}-\boldsymbol{e}_h+\boldsymbol{e}_j} a_{ij} = 0$$

The already settled case yields $\sum_{j=1}^{c} b_{\boldsymbol{s}-\boldsymbol{e}_h+\boldsymbol{e}_j} u_j \in (0 : \mathfrak{j}) U$, hence $b_{\boldsymbol{s}} \in (0 : \mathfrak{j})$. □

Proof of Theorem 4.1. In this proof $\varphi \colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, k)$ denotes a surjective homomorphism of local rings. We start with some preliminary constructions.

For the topology defined by the powers of $\operatorname{Ker} \varphi$, the completion $\widehat{R}$ of R is flat over R, and each S-module is discrete. It follows that $\operatorname{Tor}_*^{\widehat{R}}(S, S') \cong \operatorname{Tor}_*^{R}(S, S')$ for each S-algebra S', and that there are induced homomorphisms of local rings $\widehat{\psi} \colon S \to \widehat{R}$ and $\widehat{\varphi} \colon \widehat{R} \to S$ such that $\widehat{\varphi}\widehat{\psi} = \mathrm{id}_S$.

Fix $a_1, \dots, a_e$ that minimally generate $\operatorname{Ker} \varphi$, set $P = S[[x_1, \dots, x_e]]$ and let $\pi \colon P \to R$ be the surjective homomorphism with $\pi(x_i) = a_i$ for each i. Choose a minimal generating set $\boldsymbol{f} = f_1, \dots, f_c$ for $\operatorname{Ker} \pi$. These are formal power series with trivial constant terms, so $f_j = \sum_{i=1}^{e} g_{ij} x_i$ with $g_{ij} \in S[[x_1, \dots, x_e]]$; the minimality of the generating set $\{a_1, \dots, a_e\}$ implies $g_{ij}(0, \dots, 0) = a_{ij} \in \mathfrak{n}$ for all (i, j).

We are now ready to prove the equivalence of the conditions of the theorem.

(iii) $\Longrightarrow$ (ii). Assume that $\boldsymbol{f}$ is a P-regular sequence in $(x_1, \dots, x_e)^2$.

If S' is an S-algebra and $R[T, U]$ is the resolution from 4.2, then $a_{ij} = 0 \in S'$ for all (i, j), so $\operatorname{Tor}_*^R(S, S') = S'[T, U]$ as graded algebras. A minimal set of generators of $\mathfrak{n}$ together with $\{a_1, \dots, a_e\}$ minimally generate $\mathfrak{m}$, so $\operatorname{card}(T) = e = \operatorname{edim} R - \operatorname{edim} S$. Finally, $\operatorname{card}(U) = c = \dim P - \dim R = \dim S + e - \dim R$.

(ii) $\Longrightarrow$ (i) is obvious.

(i) $\implies$ (iii). The maps $P \to R \to S$ define a Jacobi-Zariski exact sequence

$$\mathrm{D}_{n+1}(S|P;k) \to \mathrm{D}_{n+1}(S|R;k) \to \mathrm{D}_n(R|P;k) \to \mathrm{D}_n(S|P;k)$$

cf. [1, (5.1)]. By flat base change [1, (4.54)], we have

$$\mathrm{D}_n(S|P;k) \cong \mathrm{D}_n(k|(P \otimes_S k);k) \qquad \text{for all} \quad n \in \mathbb{Z}\,.$$

The last module vanishes for $n \geq 2$ because the ring $P \otimes_S k \cong k[[x_1, \dots, x_e]]$ is regular, cf. [1, (6.26)]. Putting these facts together, we get

$$\mathrm{D}_{n+1}(S|R;k) \cong \mathrm{D}_n(R|P;k) \qquad \text{for} \quad n \geq 2\,,$$

By hypothesis, $\mathrm{Tor}^R_*(S,S)$ is a finitely generated algebra over S, so $\mathrm{D}_n(S|R;k) = 0$ for $n \gg 0$ by Theorem 3.4, and thus $\mathrm{D}_n(R|P;k) = 0$ for $n \gg 0$. By [6, (3.2)] the projective dimension $\mathrm{pd}_P R$ is finite, hence $\boldsymbol{f}$ is a regular sequence by 3.3.3.2.

We can now apply Proposition 4.3, whose notation we adopt. It yields a direct sum decomposition $B = C \oplus D$ of $B = \mathrm{Tor}^R_*(S,S)$, where $C = \bigoplus_{n<2\ell} B^{(\ell)}_n$ is an ideal and $D = \bigoplus_n B^{(n)}_{2n}$ is a subalgebra. The same proposition shows that $E = \bigoplus_n B^{(n+e)}_{2n+e}$ is an ideal of the graded algebra B, and $CE = 0$. By hypothesis B is finitely generated as an algebra over the noetherian ring S, hence the ideal E of B is finitely generated, and thus E is finite as a module over the algebra $B/C = D$.

The vanishing lines of $A^{(\ell)}_n$ yield exact sequences of graded S-modules

$$0 \to D \to S[U] \xrightarrow{\partial} S[U] \otimes_S ST\,;$$

$$S[U] \otimes_S {\textstyle\bigwedge}^{e-1}(ST) \xrightarrow{\partial} S[U] \otimes_S {\textstyle\bigwedge}^{e}(ST) \to E \to 0\,.$$

The map $b \in S[U] \mapsto b \cdot t_1 \cdots t_e \in S[U] \otimes_S \bigwedge^e(ST)$ is a degree e homomorphism $\tau\colon S[U] \to E$ of graded D-modules. As $\partial(S[T,U]) \subseteq \mathfrak{n}S[T,U]$, we see that

$$\tau \otimes_D k\colon S[U] \otimes_D k \to E \otimes_D k$$

is bijective. For each $n \in \mathbb{Z}$ the degree n component of the D-module $S[U]$ is a finite S-module, and vanishes for $n < 0$, so by the appropriate version of Nakayama's Lemma the D-module $S[U]$ is finite. In particular, each $u \in U$ satisfies an equation

$$u^r + z_{r-1}u^{r-1} + \cdots + z_1 u + z_0 = 0 \in S[U]$$

of integral dependence with $z_j \in D$. Differentiating one with minimal r, we get

$$\big(ru^{r-1} + (r-1)z_{r-1}u^{r-2} + \cdots + z_1\big)\partial(u) = 0 \in S[U] \otimes_S ST\,.$$

The minimality of r implies that the coefficient of $\partial(u)$ is non-zero, hence it is not a zero-divisor on the free $S[U]$-module $S[U] \otimes_S ST$, and so $\partial(u) = 0$. Thus,

$$\textstyle\sum_{i=1}^{e} a_{ij} t_i = \partial(u_j) = 0 \quad \text{for } j = 1, \dots, c$$

so all a_{ij} vanish. Since $a_{ij} = g_{ij}(0, \dots, 0)$ where $g_{ij} \in S[[x_1, \dots, x_e]]$ appear in equalities $f_j = \sum_{i=1}^{e} g_{ij} x_i$, we get $f_j \in (x_1, \dots, x_e)^2$ for $j = 1, \dots, c$, as desired. ☐

5. Hochschild homology

In this section we bring the local results of the preceding discussion to bear on the Hochschild homology of flat $\Bbbk$-algebras essentially of finite type. We start by recalling the classical interpretation of Hochschild homology as a derived functor.

5.1. Let S be a flat $\Bbbk$-algebra, set $R = S \otimes_{\Bbbk} S$ and let $\mu\colon R \to S$ be the multiplication map $\mu(a' \otimes a'') = a'a''$. The flatness hypothesis yields an isomorphism

$$\mathrm{HH}_*(S|\Bbbk) \cong \mathrm{Tor}_*^R(S,S)$$

of graded S-algebras, cf. Cartan-Eilenberg [11, §XI.6] or Loday [23, §4.2]. If $\mathfrak{n}$ is a prime ideal of S and $\mathfrak{m} = \mu^{-1}(\mathfrak{n})$, then μ induces a surjective local homomorphism $\varphi\colon R_{\mathfrak{m}} \to S_{\mathfrak{n}}$, and there are canonical isomorphisms

$$\mathrm{Tor}_*^R(S,S) \otimes_S S_{\mathfrak{n}} \cong \mathrm{Tor}_*^{S_{\mathfrak{n}} \otimes_{\Bbbk} S_{\mathfrak{n}}}(S_{\mathfrak{n}}, S_{\mathfrak{n}}) \cong \mathrm{Tor}_*^{R_{\mathfrak{m}}}(S_{\mathfrak{n}}, S_{\mathfrak{n}})\,.$$

Next we prove the theorems announced in the introduction.

5.2. Theorem. *If S is a flat commutative algebra essentially of finite type over a commutative noetherian ring $\Bbbk$, and $\mathrm{HH}_n(S|\Bbbk) = 0$ for an even positive n and an odd positive n, then S is smooth over $\Bbbk$.*

Proof. Due to the isomorphisms of 5.1, Theorem 2.1 shows that $(\mathrm{Ker}\,\mu)_{\mathfrak{m}}$ is generated by an $R_{\mathfrak{m}}$-regular sequence, so S is smooth, cf. [23, (3.4.2)]. □

5.3. Theorem. *If S is a flat commutative algebra essentially of finite type over a commutative noetherian ring $\Bbbk$, and the algebra $\mathrm{HH}_*(S|\Bbbk)$ is finitely generated over S, then S is smooth over $\Bbbk$.*

Proof. Let $\mathfrak{n}$ be a prime ideal of S, and set $k = S_{\mathfrak{n}}/\mathfrak{n}S_{\mathfrak{n}}$.

When $\mathrm{char}(k) > 0$ Theorem 3.1 and 5.1 show that $\mathrm{Ker}\,\varphi$ is generated by an $R_{\mathfrak{m}}$-regular sequence; as in the preceding proof, it follows that S is smooth.

When $\mathrm{char}(k) = 0$, consider the homomorphism $S \to S \otimes_{\Bbbk} S$ given by $a \mapsto a \otimes 1$. It localizes to a homomorphism $\psi\colon S_{\mathfrak{n}} \to R_{\mathfrak{m}}$ satisfying $\varphi\psi = \mathrm{id}_{S_{\mathfrak{n}}}$. Thus, Theorem 4.1 applies, and shows that the $S_{\mathfrak{n}}$-module $\mathrm{Tor}_1^{R_{\mathfrak{m}}}(S_{\mathfrak{n}}, S_{\mathfrak{n}}) \cong \Omega^1_{S_{\mathfrak{n}}|\Bbbk}$ is free, hence S is smooth by the Jacobian criterion, cf. [16, (17.15.8)] or [1, (7.31)]. □

Proposition 5.6 and Example 5.7 show that the homological hypothesis in the statements of the preceding theorems cannot be significantly weakened. We briefly consider relaxing the finiteness hypothesis on the $\Bbbk$-algebra S.

5.4. Remark. An attentive reader might have noticed that the preceding proofs show that S is regular over $\Bbbk$ even when the hypothesis that S is essentially of finite type over $\Bbbk$ is weakened to an assumption that $(S \otimes_{\Bbbk} S)_{\mathfrak{m}}$ is noetherian for each prime ideal $\mathfrak{m}$ in $S \otimes_{\Bbbk} S$ containing $\mathrm{Ker}\,\mu$. We have stated the results under the stronger hypothesis because it is easy to check, and because Ferrand [15, (3.6)] proves that it covers most cases: If $S_{\mathfrak{n}} \otimes_{\Bbbk} S_{\mathfrak{n}}$ is noetherian for some prime ideal $\mathfrak{n}$ of S, then $S_{\mathfrak{n}}$ is essentially of finite type over $\Bbbk$.

For complete intersections we can prove more by using a special resolution.

5.5. Let $P = \Bbbk[x_1, \dots, x_e]$ be a polynomial ring over a noetherian ring $\Bbbk$, and let $\boldsymbol{f} = f_1, \dots, f_c$ be a P-regular sequence such that $S = P/(\boldsymbol{f})$ is flat over $\Bbbk$.

5.5.1. If $\partial_i(f_j)$ is the image in S of the partial derivative $\partial f_j/\partial x_i$, then

$$\mathrm{HH}_*(S|\Bbbk) \cong \mathrm{H}_*\left(S\big\langle t_1, \dots, t_e; u_1, \dots, u_c \,\big|\, \partial(t_i) = 0\,; \partial(u_j) = \sum_{i=1}^{e} \partial_i(f_j)t_i \big\rangle\right).$$

When $\Bbbk$ contains a field of characteristic zero the isomorphism is implicit in a general theorem of Quillen [26, (8.6)]; explicitly, it appears in an argument of

Wolffhardt [33, p. 61]. Over a noetherian ring $\Bbbk$ the formula is proved by Guccione and Guccione [18, (3.2)] and by Brüderle and Kunz [9, (5.2)]; in both papers it is deduced from Tate's resolution 4.2. The isomorphism above transforms the *Hodge decomposition* of Hochschild homology, cf. [23, (4.5)], into the direct sum decomposition of the right hand side given by Proposition 4.3: this is proved by Cortiñas, Guccione, and Guccione [12, (3.4.2)].

Specialized to hypersurfaces, the formula above shows that Hochschild homology can be computed in terms of exterior powers of the module of Kähler differentials and the homology of an appropriate Koszul complex, cf. e.g. [9, (5.5)].

5.5.2. If $\boldsymbol{f} = f$ and K is the Koszul complex on $\partial_1(f), \dots, \partial_e(f) \in S$, then

$$\mathrm{HH}_n(S\,|\,\Bbbk) \cong \begin{cases} \bigwedge_S^n \Omega^1_{S\,|\,\Bbbk} \oplus \bigoplus_{i\geqslant 1} \mathrm{H}_{e+2i-n}(K) & \text{for } 0 \le n \le e\,; \\ \bigoplus_{i\geqslant 0} \mathrm{H}_{e+2i-n}(K) & \text{for } n \ge e+1\,. \end{cases}$$

As a first application we show that over any noetherian domain $\Bbbk$ that is not a field and any integer $e \geq 1$ there exist algebras for which the even part or the odd part of the Hochschild homology vanishes beyond degree e, and the other does not.

5.6. Proposition. *Let b be a non-unit non-zero-divisor in a noetherian ring $\Bbbk$.*

For each $e \geq 1$ there exist pairwise coprime positive integers $a_1, \dots, a_e$ such that neither $\Bbbk$ nor $\Bbbk/(b)$ has additive torsion of order a_i for $i = 1, \dots, e$. Furthermore, for such integers a_i and b the $\Bbbk$-algebra $S = \dfrac{\Bbbk[x_1, \dots, x_e]}{(x_1^{a_1} + \dots + x_e^{a_e} + b)}$ satisfies $\mathrm{HH}_n(S\,|\,\Bbbk) \neq 0$ if and only if $0 \leq n \leq e$ or $n = e + 2i$ for some $i > 0$.

Proof. By hypothesis, the ring $T = \Bbbk/(b) \cong S/(x_1, \dots, x_e)$ is not trivial. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated prime ideals of the $\Bbbk$-module $\Bbbk \oplus T$, and let $p_j \in \mathbb{Z}$ be the natural number that generates $\mathfrak{p}_j \cap \mathbb{Z}$. Choose $a_1, \dots, a_e \in \mathbb{N}$ whose prime decompositions involve only primes from pairwise non-intersecting subsets in the complement of $\{p_1, \dots, p_s\}$; these integers have the desired property.

The isomorphism $\Omega^1_{S\,|\,\Bbbk} \cong S^e/(a_1x_1^{a_1-1}, \dots, a_ex_e^{a_e-1})$ induces isomorphisms

$$\textstyle\bigwedge_S^n(\Omega^1_{S\,|\,\Bbbk}) \otimes_S T \cong \bigwedge_T^n(\Omega^1_{S\,|\,\Bbbk} \otimes_S T) \cong \bigwedge_T^n(T^e)$$

for all $n \in \mathbb{Z}$; therefore, $\bigwedge_S^n \Omega^1_{S\,|\,\Bbbk} \neq 0$ for $0 \leq n \leq e$. In view of 5.5.2, it remains to prove that the Koszul complex $K = S\langle v_1, \dots, v_e \,|\, \partial(v_i) = a_ix_i^{a_i-1}\rangle$ has no homology in positive degrees. Setting $P = \Bbbk[x_1, \dots, x_e]$ and $f = x_1^{a_1} + \dots + x_e^{a_e} + b$, we have $K \cong P\langle v_1, \dots, v_e \,|\, \partial(v_i) = a_ix_i^{a_i-1}\rangle \otimes_P S$, so by Lemma 1.3 it suffices to show that

$$\mathrm{H}_n(P\langle v_1, \dots, v_e, v \,|\, \partial(v_i) = a_ix_i^{a_i-1}, \partial(v) = f\rangle) = 0 \quad \text{for} \quad n \neq 0\,.$$

This holds because the sequence $a_1x_1^{a_1-1}, \dots, a_ex_e^{a_e-1}, f$ is P-regular. □

The next application shows that if S is not smooth over $\Bbbk$, then the Hochschild homology algebra need not be finitely generated even as a Γ-algebra, cf. 2.4.

5.7. Example. If $\Bbbk$ is a field of characteristic $p > 0$, and $S = \Bbbk[x]/(x^d)$ for an integer $d \geq 2$ that is not divisible by p, then for all $i \geq 1$ the S-module $\mathrm{HH}_{2i}(S\,|\,\Bbbk)$ is generated by $\mathrm{cls}(xu^{(i)})$. The definition and properties of divided powers in $\mathrm{HH}_*(S\,|\,\Bbbk)$ show that $\mathrm{cls}(xu^{(i)})^{(d)} = x^d \,\mathrm{cls}(u^{(i)})^{(d)} = 0$.

6. Nilpotence

In this section we study nilpotence properties of shuffle products in Hochschild homology. Our main result in this direction significantly generalizes [32, (2.7)], where it is assumed that S is graded and finite over $S_0 = \Bbbk$.

6.1. Theorem. *Let $\Bbbk$ be a field, and S a $\Bbbk$-algebra essentially of finite type that is locally complete intersection. If $\operatorname{char}(\Bbbk) = 0$, or if S is reduced and for each minimal prime ideal $\mathfrak{q}$ of S the field extension $\Bbbk \subseteq S_{\mathfrak{q}}$ is separable, then*

$$(\mathrm{HH}_{\geqslant 1}(S|\Bbbk))^s = 0 \quad \textit{for some integer} \quad s \geq 1\,.$$

Proof. First we treat a special case: $S = P/(\boldsymbol{f})$ satisfies the hypotheses of 5.5.1.

Proposition 4.3.1 then yields $B = C \oplus D$ with $C = \bigoplus_{n<2m} B_n^{(\ell)}$ and $D = \bigoplus_n B_{2n}^{(n)}$, and shows that $C^{e+1} \subseteq \bigoplus_{n+e<2\ell} B_n^{(\ell)}$. By Proposition 4.3.2 the last module is trivial, so it remains to prove that $D_{\geqslant 1}$ is nilpotent. Proposition 4.3.1 yields $D_{\geqslant 1} \subseteq \bigoplus_{n\geqslant 1}(0 : \mathfrak{j})A_{2n}^{(n)}$, where $\mathfrak{j}$ is the ideal in S generated by the $c \times c$ minors of the Jacobian matrix $\big(\partial f_j/\partial x_i\big)$, so we show that $(0 : \mathfrak{j})$ is nilpotent. If $\operatorname{char}(\Bbbk) = 0$, then Eisenbud, Huneke, and Vasconcelos [14, (2.2)] prove that $(0 : \mathfrak{j})$ is the nilradical of S. If S is reduced and generically smooth over $\Bbbk$, then the linear map $S^c \to S^e$ given by the Jacobian matrix is injective, hence $(0 : \mathfrak{j}) = 0$.

Next we turn to the general case: S is a localization of a residue ring of a polynomial ring P over $\Bbbk$. Fix $\mathfrak{n} \in \operatorname{Spec}(S)$, and let $\mathfrak{m}$ be its inverse image in P; by hypothesis $S_{\mathfrak{n}} = P_{\mathfrak{m}}/(\boldsymbol{f})$ where $\boldsymbol{f}$ is a $P_{\mathfrak{m}}$-regular sequence that we may take in P.

The Koszul complex $K = P\langle T|\partial(T) = \boldsymbol{f}\rangle$ satisfies $\mathrm{H}_1(K)_{\mathfrak{m}} \cong \mathrm{H}_1(K_{\mathfrak{m}}) = 0$, so we can find $h \in P \smallsetminus \mathfrak{m}$ with $h\,\mathrm{H}_1(K) = 0$. The isomorphism $P' \cong P[y]/(hy-1)$ identifies $\operatorname{Spec}(P')$ with the open set $D(h) = \{\mathfrak{p} \in \operatorname{Spec} P | h \notin \mathfrak{p}\}$ of $\operatorname{Spec} P$. For each $\mathfrak{p} \in D(h)$ we have $\mathrm{H}_1(K \otimes_R P'_{\mathfrak{p}}) = 0$, so the sequence $\boldsymbol{f}$ is $P'_{\mathfrak{p}}$-regular, and hence the ideal $(\boldsymbol{f})P'$ can be generated by a P'-regular sequence. If $\boldsymbol{g}$ is a lifting of such a sequence in the polynomial ring $P[y]$, then $S_{\mathfrak{n}}$ is a localization of the $\Bbbk$-algebra $S_h \cong P[y]/(hy-1, \boldsymbol{g})$, where $hy-1, \boldsymbol{g}$ is a $P[y]$-regular sequence. Hochschild homology algebras commute with localization, so for each s we have

$$(\mathrm{HH}_{\geqslant 1}(S|\Bbbk)^s)_h \cong (\mathrm{HH}_{\geqslant 1}(S_h|\Bbbk))^s\,.$$

The first part of the proof shows that the right hand side vanishes for $s \gg 0$. The open sets $D(h) \cap \operatorname{Spec} S$ cover $\operatorname{Spec} S$, so by quasi-compactness we can find $h_1, \dots, h_t$ such that $\operatorname{Spec} S = \bigcup_{i=1}^t D(h_i)$. For large enough s we have $(\mathrm{HH}_{\geqslant 1}(S_{h_i}|\Bbbk))^s = 0$ for $i = 1, \dots, t$, hence we conclude that $(\mathrm{HH}_{\geqslant 1}(S|\Bbbk))^s = 0$, as desired. □

The next example shows that, in general, Hochschild homology is not *nilpotent*; it also serves to illustrate that the hypotheses of the theorem above are sharp.

6.2. Example. If $\Bbbk$ is a field of characteristic $p > 0$ and $S = \Bbbk[x]/(x^p - a)$ with $a \notin \Bbbk^p$, then for the purely inseparable field extension $\Bbbk \subseteq S$ we have

$$\mathrm{HH}_*(S|\Bbbk) = \mathrm{H}_*\left(\, S\langle t, u|\partial(t) = 0; \partial(u) = 0\rangle \,\right) = S\langle t, u\rangle$$

by 5.5.1, and the product rule in 2.4 yields $u \cdot u^{(p)} \cdots u^{(p^s)} \neq 0$ for all $s \geq 1$.

Nevertheless, when $\Bbbk$ is a field of characteristic $p > 0$ the ideal $\mathrm{HH}_{\geqslant 1}(S|\Bbbk)$ is *nil* of exponent p, due to the following immediate consequence of 5.1 and 3.5.

6.3. Remark. If there is a positive integer q such that $qS = 0$, then $w^q = 0$ for each homology class $w \in \mathrm{HH}_{\geqslant 1}(S|\Bbbk)$.

In view of the preceding theorem and remark, in an earlier version of this paper we raised the question whether Hochschild homology over a field of characteristic 0 is nil. Löfwall and Sköldberg, and independently Larsen and Lindenstrauss, answered our question in the negative for the test algebra that we suggested. With their permission, we include the argument of Larsen and Lindenstrauss.

6.4. Proposition. *If* $\Bbbk$ *is a field of characteristic* 0 *and* $S = \Bbbk[x,y]/(x^2,xy,y^2)$, *then there is a class* $\operatorname{cls}(z) \in \operatorname{HH}_4(S|\Bbbk)$ *such that* $\operatorname{cls}(z)^n \neq 0$ *for each* $n \geq 1$.

Proof. We set $\mathfrak{n} = (x,y) \subseteq S$ and denote $\otimes$ tensor products over $\Bbbk$. The Hochschild complex C of the $\Bbbk$-algebra S has degree n component $C_n = S \otimes \mathfrak{n}^{\otimes n}$ and differential

$$\partial(s \otimes a_1 \otimes \cdots \otimes a_n) = (sa_1) \otimes a_2 \otimes \cdots \otimes a_n + (-1)^n (a_n s) \otimes a_1 \otimes \cdots \otimes a_{n-1}\,,$$

due to the equality $\mathfrak{n}^2 = 0$, cf. [11, §IX.6] or [23, §1.1]. In particular, $\partial(C) \subseteq \mathfrak{n}C$, so if z is a cycle and $z^n \notin \mathfrak{n}C$, then $\operatorname{cls}(z)^n \neq 0$. A direct computation shows that

$$z = 1 \otimes (x \otimes x \otimes y \otimes y - y \otimes x \otimes x \otimes y + y \otimes y \otimes x \otimes x - x \otimes y \otimes y \otimes x) \in C_4$$

is a cycle. Denote c_n the coefficient with which the tensor monomial

$$v_n = 1 \otimes \underbrace{x \otimes \cdots \otimes x}_{2n} \otimes \underbrace{y \otimes \cdots \otimes y}_{2n} \in C_{4n} = S \otimes \mathfrak{n}^{\otimes 4n}$$

appears in z^n. By the definition of shuffle product, cf. [11, §XI.6] or [23, §4.2], any monomial occuring in a product involving one of the elements $1 \otimes y \otimes x \otimes x \otimes y$, $1 \otimes y \otimes y \otimes x \otimes x$, or $1 \otimes x \otimes y \otimes y \otimes x$ contains $y \otimes x$ as a submonomial, so c_n is equal to the coefficient of v_n in $(1 \otimes x \otimes x \otimes y \otimes y)^n$. It is clear that $c_1 = 1$, so we assume that $c_{n-1} = ((n-1)!)^2$ for some integer $n \geq 2$. Note that $c_n = b_n c_{n-1}$, where b_n is the coefficient with which v_n appears in $v_1 \cdot v_{n-1}$, and that each such appearance comes from a permutation ξ that shuffles the x's separately from the y's. Thus, if Ξ denotes the set of $(2, 2n-2)$-shuffles, then $b_n = \big(\sum_{\xi\in\Xi} \operatorname{sign}(\xi)\big)^2$. The equalities

$$\sum_{\xi\in\Xi} \operatorname{sign}(\xi) = \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n} (-1)^{i+j-3} = \underbrace{1 + 0 + 1 + 0 + \cdots + 1}_{2n-1} = n\,,$$

yield $c_n = (n!)^2 \neq 0 \in \Bbbk$, hence $z^n \notin \mathfrak{n}C$ for each $n \geq 0$, as desired. □

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