

FACTORIAL EXTENSIONS OF REGULAR LOCAL RINGS AND INVARIANTS OF FINITE GROUPS

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Dedicated to the memory of Maurice Auslander

INTRODUCTION

It is often useful to approach the singularity of a noetherian commutative *local domain* S from a *regular local subring* P over which S is *finite* as a module and *residually trivial* (that is, the induced extension of residue fields is trivial). Such subrings exist in a variety of situations: in commutative algebra they are given by Cohen’s structure theory for complete local rings; in analytic geometry they are in place by definition; in algebraic geometry they are sometimes produced by Noether normalization.

If S is *normal* (here meaning noetherian and integrally closed) and divisorially unramified over P , then $S = P$ by the classical purity property of the branch locus, cf. [2]. Thus, when studying finite normal local extensions of a regular local ring P a natural next step is to allow a controlled amount of ramification in codimension one.

An early example appears in [15], where Serre uses purity to show that if $P \hookrightarrow S$ is ramified in codimension one only at principal primes of S , and is *generically Galois* in the sense that the induced extension of fraction fields $P_0 \hookrightarrow S_0$ is Galois, then $\text{Gal}(S_0|P_0)$ is generated by generalized reflections (an automorphism h of S is a generalized reflection if $(h - \text{id})(S) \subseteq (x)$ for some non-unit $x \in S$).

The preceding result is applicable, in particular, when S is *factorial*. Recently Griffith has obtained in [9] structure theorems for factorial domains, with the condition on the fraction field extension weakened to one of ‘uniform’ divisorial ramification. His results assume that the regular ring P is complete with algebraically closed residue field of characteristic zero, and his arguments strongly depend on these additional conditions.

This raises the question of describing the structure of local factorial extensions of more general regular local rings. Our hypotheses appear in a diagram of inclusions of local rings

$$\begin{array}{ccccccc}
 & & \overbrace{\hspace{10em}}^{\text{residually trivial and generically Galois}} & & & & \\
 \frac{1}{n} \in & P & \xrightarrow{\text{generically Galois}} & D & \xrightarrow{\text{divisorially unramified}} & S & \longrightarrow E \ni \sqrt[n]{1} \\
 \text{regular} & & & \text{normal} & & \text{factorial} & \text{normal}
 \end{array}$$

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where $n = \text{rank}_P E$ and $\sqrt[n]{1}$ denotes a primitive n 'th root of unity.

Denoting by Q (respectively, R) the integral closure of P in the *largest abelian* (respectively, *largest Galois*) extension of P_0 contained in S_0 , and by T its integral closure in the *smallest Galois* extension of P_0 containing S_0 , we extend and sharpen Griffith's results by exhibiting a *canonical splitting of the divisorial ramification* of S over P into a sequence of 'nice' intermediate extensions. It is described by a second picture

$$\begin{array}{ccccccc}
 & & \text{Galois group generated by} & & & & \\
 & & \text{generalized reflections} & & & & \\
 & \overbrace{\hspace{10em}} & & \overbrace{\hspace{10em}} & & & \\
 P & \xrightarrow{\text{Kummer}} & Q & \xrightarrow{\text{divisorially}} & R & \xrightarrow{\text{divisorially}} & S & \xrightarrow{\text{divisorially}} & T \\
 & & \text{normal} & & \text{factorial} & & & & \text{normal} \\
 & & \text{complete} & & \text{complete} & & & & \\
 & & \text{intersection} & & \text{intersection} & & & &
 \end{array}$$

where a Kummer extension (or Kummer covering, cf. Grothendieck and Murre [10]) is a tamely ramified abelian extension, obtained by adjunction of suitable roots.

The extensions $P \hookrightarrow R$ and $R \hookrightarrow T$ are handled in very different ways.

In view of Serre's result, the first inclusion is studied in the framework of a problem of invariant theory: the action on a local domain (R, \mathfrak{m}, k) of a finite group G generated by generalized reflections, whose order is invertible in R . In Section 1 we determine the invariants of the commutator subgroup G' for the induced action on the symmetric algebra of the k -vector space $\mathfrak{m}/\mathfrak{m}^2$; on it G acts as a linear group generated by pseudo-reflections, so we have at our disposal an attractive chapter of classical invariant theory. In Section 2 we use this explicit computation together with the comprehensive local results of Singh [16] and Avramov [4] to prove that Q is a Kummer extension of P . When, in addition, R is integrally closed, we establish in Section 3 that Q is divisorially unramified over P .

The factorial ring S appears in Section 4, where we show that R is factorial and $R \hookrightarrow T$ is divisorially unramified by combining arguments from [9] with work in preceding sections.

Finally, a note on terminology. We say that a ring extension is *normal* if both rings are integrally closed noetherian domains. To avoid confusion with other uses of 'normal extension', we adopt Bourbaki's term *quasi-Galois extension* for the inclusion of a field into the splitting field of a polynomial; of course, the prefix 'quasi' is dropped when the field extension is separable. For background in ramification theory we refer to Abhyankar [1], or to Auslander and Buchsbaum [3].

1. POLYNOMIAL INVARIANTS

In this section we focus on a finite group $G \leq \text{GL}_k(V)$ acting on a finite k -vector space V , and assume that the order n of G is invertible in k . An element $h \in G$ is a *pseudo-reflection* if $\text{rank}_k(h - \text{id}) = 1$. We denote by \mathcal{P} the set of pseudo-reflections of G , and with each $h \in \mathcal{P}$ we associate the following objects:

- $L[h] := \text{Im}(h - \text{id}) \subseteq V$, the *reflected line* of h .
- $H[h] := \text{Ker}(h - \text{id}) \subseteq V$, the *reflecting hyperplane* of h .
- $G[h] := \{g \in G \mid g(v) = v \text{ for all } v \in H[h]\} \leq G$, the *stabilizer* of $H[h]$.

Clearly, G acts on \mathcal{P} by conjugation. This induces actions on the sets $\mathcal{L} = \{L[h]\}_{h \in \mathcal{P}}$ and $\mathcal{H} = \{H[h]\}_{h \in \mathcal{P}}$ by translation: $g(L[h]) = L[ghg^{-1}]$ and $g(H[h]) = H[ghg^{-1}]$, and on $\mathcal{G} = \{G[h]\}_{h \in \mathcal{P}}$ by conjugation: $g(G[h])g^{-1} = G[ghg^{-1}]$.

It follows from [4], (13), that the natural maps $\mathcal{L} \leftarrow \mathcal{P} \rightarrow \mathcal{H} \rightarrow \mathcal{G}$ induce G -equivariant bijections $\mathcal{L} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{G}$. In particular, there is a bijective correspondence $\mathcal{L}_i \leftrightarrow \mathcal{G}_i$ between the orbits $\mathcal{L}_1, \dots, \mathcal{L}_s$ and $\mathcal{G}_1, \dots, \mathcal{G}_s$ of the actions of G on the sets \mathcal{L} and \mathcal{G} . It follows from Maschke's theorem that each $G[h]$ is cyclic and its non-identity elements are pseudo-reflections. For $i = 1, \dots, s$ we choose in \mathcal{G}_i a cyclic group $G_i = \langle h_i \rangle$ and set $n_i = |G_i|$.

Proposition 1. *Let V be a finite vector space over a field k , let $G \leq \mathrm{GL}_k(V)$ be a finite group with order invertible in k , and consider the natural actions of G and of its commutator subgroup G' on the symmetric algebra $C = \mathrm{Sym}_k(V)$.*

Let Y_1, \dots, Y_s be variables over $A = C^G$ of degree $|\mathcal{L}_1|, \dots, |\mathcal{L}_s|$, respectively, and consider the action of the group $\prod_{i=1}^s G_i$ on $A[Y_1, \dots, Y_s]$ by A -algebra automorphisms defined by $(g_1, \dots, g_s)(Y_i) = \det(g_i)Y_i$. For each pseudo-reflection $h \in \mathcal{P}$ choose a non-zero vector $e[h] \in L[h]$, and for $i = 1, \dots, s$ set $e_i = \prod_{L[h] \in \mathcal{L}_i} e[h] \in C$.

If G is generated by pseudo-reflections, then the elements e_i are in $B = C^{G'}$, the elements $f_i = e_i^{n_i}$ are in A , and the degree zero homomorphism of graded A -algebras $A[Y_1, \dots, Y_s] \rightarrow C$ which maps Y_i to e_i induces a ρ -equivariant isomorphism

$$\varphi: \frac{A[Y_1, \dots, Y_s]}{(Y_1^{n_1} - f_1, \dots, Y_s^{n_s} - f_s)} \cong B$$

where $\rho: \prod_{i=1}^s G_i \rightarrow G_{\mathrm{ab}}$ is a group isomorphism given by $\rho(g_1, \dots, g_s) = (g_1 \cdots g_s)G'$.

Remark. The k -algebra A is generated by $\mathrm{rank}_k V$ algebraically independent forms by the Chevalley–Shephard–Todd Theorem, cf. [6], Theorem 4 or [5], Theorem 7.2.1.

Proof. Dualizing [17], Theorem 4.3.4, or [18], Theorem 3.1, for each $h \in \mathcal{P}$ we get

$$h(e_i) = \begin{cases} \det(h)e_i & \text{if } L[h] \in \mathcal{L}_i ; \\ e_i & \text{if } L[h] \notin \mathcal{L}_i . \end{cases}$$

As $\det(h)$ is a primitive root of unity of order $|h|$, direct computations yield $g(f_i) = f_i$ and $ghg^{-1}h^{-1}(e_i) = e_i$ for all $g, h \in G$, so f_i is G -invariant and e_i is G' -invariant for $i = 1, \dots, s$. The formulas also show that the homomorphism φ is ρ -equivariant.

For each $h \in \mathcal{P}$, there exist $g \in G$ and $i \in \{1, \dots, s\}$ such that $gh^{-1}g^{-1} \in G_i$, hence h is congruent modulo G' to a power of h_i . Thus, h_1G', \dots, h_sG' generate G_{ab} , so ρ is surjective. This observation produces the inequality below

$$\prod_{i=1}^s n_i = \left| \prod_{i=1}^s G_i \right| \geq |G_{\mathrm{ab}}|$$

and shows that if equality holds, then ρ is bijective.

In view of the transitivity formulas $A = C^G = (C^{G'})^{G/G'} = B^{G_{\text{ab}}}$, Galois theory yields

$$|G_{\text{ab}}| = \text{rank}_A B.$$

The finite extension B of the graded polynomial ring A is a Cohen–Macaulay ring by [11], Proposition 15, and thus a free A -module. It follows that

$$\text{rank}_A B = \text{rank}_k \overline{B}$$

where an overbar denotes the result of tensoring over A with $k = A/A_+$.

For the composition of φ with the injection of A -algebras $\iota: B \rightarrow C$ we have

$$(\iota\varphi)\left(\prod_{i=1}^s Y_i^{n_i-1}\right) = \prod_{i=1}^s e_i^{n_i-1} = \prod_{i=1}^s \prod_{L[h] \in \mathcal{L}_i} e[h]^{n_i-1} = \prod_{i=1}^s \prod_{G[h] \in \mathcal{G}_i} \prod_{\substack{g \in G[h] \\ g \neq \text{id}}} e[g] = \prod_{h \in \mathcal{P}} e[h].$$

It is shown in [4], §5, Step 4, that the image of the last element in \overline{C} is not zero, hence $y = \prod_{i=1}^s \overline{Y}_i^{n_i-1}$ is not in the kernel of $\overline{\iota\varphi}$. As y generates the socle of the artinian complete intersection $k[Y_1, \dots, Y_s]/(Y_1^{n_1}, \dots, Y_s^{n_s})$, it follows that $\overline{\iota\varphi}$ is injective. Thus, $\overline{\varphi}$ is injective, and as a consequence

$$\text{rank}_k \overline{B} \geq \text{rank}_k \frac{k[Y_1, \dots, Y_s]}{(Y_1^{n_1}, \dots, Y_s^{n_s})} = \prod_{i=1}^s n_i.$$

Putting together the (in)equalities obtained above, we get equalities

$$\prod_{i=1}^s n_i = |G_{\text{ab}}| \quad \text{and} \quad \text{rank}_k \frac{k[Y_1, \dots, Y_s]}{(Y_1^{n_1}, \dots, Y_s^{n_s})} = \text{rank}_k \overline{B}.$$

As already noted, the first one shows that ρ is bijective. In view of the injectivity of $\overline{\varphi}$, the second one implies that $\overline{\varphi}$ is bijective. Since $\overline{\varphi} = k \otimes_A \varphi$, and φ is an A -linear homomorphism of graded free A -modules, it is an isomorphism by Nakayama. \square

2. LOCAL INVARIANTS

In this section G is a finite subgroup of the group of automorphisms of a local domain (R, \mathfrak{m}, k) which induce the trivial action on k , and we assume that the order of G is invertible in R . An element $h \in G$ is a *generalized reflection* if $h \neq \text{id}$ and there exists a principal ideal $\mathfrak{a} \neq R$ such that $h(a) - a \in \mathfrak{a}$ for all $a \in R$. We know by [12], Corollary 3.4, that R is finite over R^G , by [16], Lemma 3, that the natural homomorphism of groups $\varepsilon: G \rightarrow \text{GL}_k(\mathfrak{m}/\mathfrak{m}^2)$ is injective, and by [4], (12), that h is a generalized reflection if and

only if $\varepsilon(h)$ is a pseudo-reflection. The ideal \mathfrak{a} in the definition of a generalized reflection is uniquely defined as the *inertia ideal* $\mathfrak{i}[h]$, generated by $\{h(a) - a \mid a \in R\}$, cf. [4], (12).

We identify through ε the set of generalized reflections in G and the set of pseudo-reflections in $\varepsilon(G)$, and write \mathcal{P} for either set. For each $h \in \mathcal{P}$ the *inertia subgroup* $\{g \in G \mid g(a) - a \in \mathfrak{i}[h] \text{ for all } a \in R\}$ corresponds through ε to the stabilizer of the reflecting hyperplane of h , so we denote it by $G[h]$. The group G acts by conjugation on the set \mathcal{P} of generalized reflections. There are induced actions by conjugation on the family $\mathcal{G} = \{G[h]\}_{h \in \mathcal{P}}$ and by translation $g(\mathfrak{i}[h]) = \mathfrak{i}[ghg^{-1}]$ on the family $\mathcal{I} = \{\mathfrak{i}[h]\}_{h \in \mathcal{P}}$.

By [4], (13), there is a G -equivariant bijection between these families, and thus a bijection of orbits $\{\mathcal{G}_1, \dots, \mathcal{G}_s\} \leftrightarrow \{\mathcal{I}_1, \dots, \mathcal{I}_s\}$. As before, we choose $G_i \in \mathcal{G}_i$ and set $n_i = |G_i|$.

Theorem 2. *Let G be a group of automorphisms of a local domain (R, \mathfrak{m}, k) which induce the identity on k , and let $G' \leq G$ be the commutator subgroup; thus, $Q = R^{G'}$ is the integral closure of $P = R^G$ in the largest abelian extension of P_0 in R_0 .*

If G is generated by generalized reflections, $|G|$ is invertible in P , and R has a primitive root of unity of order equal to the least common multiple of $\{|h|\}_{h \in \mathcal{P}}$, then for $i = 1, \dots, s$ there exists a generator $x_i \in Q$ of the ideal $\prod_{\mathfrak{i}[h] \in \mathcal{I}_i} \mathfrak{i}[h] \subseteq R$ and there exists a group homomorphism $\omega_i: G \rightarrow P^\times$ such that $g(x_i) = \omega_i(g)x_i$.

Let Y_1, \dots, Y_s be indeterminates over P , let $\prod_{i=1}^s G_i$ act on $P[Y_1, \dots, Y_s]$ by P -algebra automorphisms such that $(g_1, \dots, g_s)(Y_i) = \omega_i(g_i)Y_i$.

The elements $y_i = x_i^{n_i}$ are in P for $i = 1, \dots, s$, and the homomorphism of P -algebras $P[Y_1, \dots, Y_s] \rightarrow R$ which maps Y_i to x_i induces a ρ -equivariant isomorphism of local rings

$$\psi: \frac{P[Y_1, \dots, Y_s]}{(Y_1^{n_1} - y_1, \dots, Y_s^{n_s} - y_s)} \cong Q$$

where $\rho: \prod_{i=1}^s G_i \rightarrow G_{\text{ab}}$ is a group isomorphism given by $\rho(g_1, \dots, g_s) = (g_1 \cdots g_s)G'$.

The proof uses a modest amount of Galois cohomology, for which we set the stage.

Let (R, \mathfrak{m}, k) be a local ring, let R^\times denote its group of units and let H be a finite group acting on R by ring automorphisms which induce the identity on k . The 1-cocycles $Z^1(H, R^\times)$ are the maps $\tau: H \rightarrow R^\times$ such that $\tau(gh) = \tau(g)g(\tau(h))$, the 1-coboundaries $B^1(H, R^\times)$ are the cocycles of the form $g \mapsto b^{-1}g(b)$ for some $b \in R^\times$, and $H^1(H, R^\times) = Z^1(H, R^\times)/B^1(H, R^\times)$ is the first cohomology group of H with coefficients in R^\times .

Lemma 3. *For R and H as above, the exact sequence of multiplicative groups*

$$1 \rightarrow 1 + \mathfrak{m} \rightarrow R^\times \xrightarrow{\varkappa} k^\times \rightarrow 1$$

induces a homomorphism $\varkappa_: H^1(H, R^\times) \rightarrow \text{Hom}(H, k^\times)$ with $\varkappa_*(\text{cls}(\tau)) = \varkappa\tau$.*

If $|H|$ is invertible in R , then \varkappa_ is injective.*

If furthermore R has a primitive root of unity of order equal to the exponent of the group H_{ab} , then \varkappa_* is an isomorphism, and for each $\tau \in Z^1(H, R^\times)$ there exists $b \in R^\times$ such that $\omega(g) = b^{-1}g(b)\tau(g)$ defines a homomorphism of groups $H \rightarrow (R^H)^\times$.

Proof. The injectivity of \varkappa_* is well known, cf. [16], Lemma 2.

Next we assume that R has the appropriate primitive root of unity, and show that each homomorphism $\chi : H \rightarrow k^\times$ lifts to a homomorphism $\omega : H \rightarrow (R^H)^\times$. Note that $\text{Im } \chi \in k^\times$ is cyclic, generated by a primitive root of unity of order $m = |\text{Im } \chi|$. As χ factors through H_{ab} , R has a primitive m 'th root of unity ζ . By [16], Lemma 4, this root is actually in R^H , and $\varkappa(\zeta) \in k$ is a primitive m 'th root of unity. Thus, $\varkappa(\zeta)$ generates $\text{Im } \chi$, so one can find a homomorphism $\omega : H \rightarrow (R^H)^\times$ with $\chi = \varkappa\omega$.

The composition of ω with the inclusion $(R^H)^\times \subseteq R^\times$ is clearly in $Z^1(H, R^\times)$. It follows that that \varkappa_* is surjective, and thus an isomorphism.

Finally, consider a $\tau \in Z^1(H, R^\times)$. By the surjectivity of \varkappa_* there is a $\sigma \in Z^1(H, R^\times)$ with $\text{Im } \sigma \subseteq (R^H)^\times$ and $\varkappa\tau = \varkappa\sigma$. The injectivity of \varkappa_* implies that τ and σ differ by a coboundary, that is, $\sigma(g) = b^{-1}g(b)\tau(g)$ for some $b \in R^\times$ and all $g \in H$. \square

Proof of Theorem 2. The bijectivity of ρ is obtained from that of the similarly named map in Proposition 1, by means of the isomorphism ε .

The principal ideal $\prod_{i[h] \in \mathcal{I}_i} i[h] \subseteq R$ is stable for the action of G . Thus, if a_i is a generator, then $g(a_i) \in R^\times a_i$. It is well known and easily verified that $g \mapsto a_i^{-1}g(a_i)$ defines a 1-cocycle in $Z^1(G, R^\times)$. By Lemma 3 there exist a unit b_i in R and a homomorphism $\omega_i : G \rightarrow P^\times$, such that $x_i = a_i b_i$ satisfies $g(x_i) = \omega_i(g)x_i$ for all $g \in G$. It is clear that x_i generates $\prod_{i[h] \in \mathcal{I}_i} i[h]$, and it is immediately checked that x_i is in $Q = R^{G'}$, that $x_i^{n_i}$ is in $P = R^G$, and that the homomorphism of algebras ψ is ρ -equivariant.

Along with the action of G on R we consider its induced actions on the graded k -algebras $C = \text{Sym}_k(V)$ and $\text{gr}(R)$, where $V = \mathfrak{m}/\mathfrak{m}^2$ and gr denotes an associated graded ring for the \mathfrak{m} -adic filtration. The natural surjection $C \rightarrow \text{gr}(R)$ of graded k -algebras is G -linear, and hence sends C_+^G to $(\text{gr}_+(R))^{G'}$. It is easily seen that $(\text{gr}_+(R))^{G'}$ is contained in the kernel of the induced homomorphism $\text{gr}(R) \rightarrow \text{gr}(R/\mathfrak{m}^G R)$, cf. [4], (11), for details, and thus there are natural surjective G -linear homomorphisms of graded k -algebras

$$\overline{C} = \frac{C}{(C_+^G)C} \twoheadrightarrow \frac{\text{gr}(R)}{(\text{gr}_+(R))^{G'} \text{gr}(R)} \twoheadrightarrow \text{gr}\left(\frac{R}{\mathfrak{m}^G R}\right).$$

By [4], (16), these maps are in fact bijective, and so induce an isomorphism $\overline{C}^{G'} \cong \text{gr}(R/\mathfrak{m}^G R)^{G'}$. As $|G'|$ is invertible in R , the formation of G' -invariants is an exact functor on the category of modules over the group ring $R[G']$. Thus, denoting by gr' an associated graded ring for the filtration induced from an \mathfrak{m} -adic filtration of an ambient ring, we get

$$(\text{gr}(R/\mathfrak{m}^G R))^{G'} \cong \text{gr}'((R/\mathfrak{m}^G R)^{G'}) \cong \text{gr}'(Q/\mathfrak{m}^G Q).$$

Let \bar{x}_i stand for the image of x_i in $Q/\mathfrak{m}^G Q$, write x_i^* for the initial form of \bar{x}_i in $\text{gr}'(Q/\mathfrak{m}^G Q)$, and note that x_i^* is a non-zero multiple of the form e_i introduced in Proposition 1. That result yields an isomorphism of graded k -algebras

$$\frac{k[Y_1, \dots, Y_s]}{(Y_1^{n_1}, \dots, Y_s^{n_s})} \xrightarrow{\cong} \text{gr}'\left(\frac{Q}{\mathfrak{m}^G Q}\right)$$

whose image is equal to $k[x_1^*, \dots, x_s^*]$. We conclude first that $k[\bar{x}_1, \dots, \bar{x}_s] = Q/\mathfrak{m}^G Q$, and then by Nakayama that $P[x_1, \dots, x_s] = Q$.

Thus, the homomorphism ψ is surjective. Its source is a free P -module of rank $\prod_{i=1}^s n_i$, and its target is a finite torsion-free P -module of rank $|G_{\text{ab}}|$. We see that $\psi \otimes_P P_0$ is a surjective homomorphism of P_0 -vector spaces of the same rank, and so is injective. It follows that ψ is injective, and hence an isomorphism. \square

3. NORMAL EXTENSIONS

Given a finite extension $P \hookrightarrow R$ of integrally closed domains, there is a bijective correspondence between the subfields of R_0 which contain P_0 , and the integrally closed subrings of R which contain P . In particular, if P is noetherian and R_0 is Galois over P_0 with Galois group G , then the usual Galois correspondence restricts to a bijection from the subgroups of G to the normal domains Q intermediate between P and R . Note that the group $\text{Gal}(R_0|P_0)$ operates on R by ring automorphisms, and its fixed ring is Q .

Recall that a finite ring extension $P \hookrightarrow R$ is *unramified* at a prime $\mathfrak{q} \subset R$ if $\mathfrak{q}R_{\mathfrak{q}} = \mathfrak{p}R_{\mathfrak{q}}$ for $\mathfrak{p} = \mathfrak{q} \cap P$ and the extension of residue fields $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}} \hookrightarrow R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$ is separable. In codimension one the ramification of R over P is described by the (Dedekind) *different* $\mathfrak{D}_{R|P}$, a divisorial ideal such that a height one prime $\mathfrak{q} \subseteq R$ is ramified over P if and only if it contains $\mathfrak{D}_{R|P}$, cf. [5], Theorem 3.10.2. Thus, $Q \hookrightarrow R$ is *divisorially unramified* (that is, unramified at all height one primes of R) if and only if $\mathfrak{D}_{R|P} = R$, cf. [3], Proposition 3.6.

Theorem 4. *If in Theorem 2 the ring R is normal, then $Q \hookrightarrow R$ is divisorially unramified.*

Proof. We use the notation of Theorem 2 and its proof. In particular, we recall that

$$R x_i = \prod_{i[h] \in \mathcal{I}_i} i[h] \text{ for } i = 1, \dots, s, \text{ and set } z = \prod_{i=1}^s x_i^{n_i-1}.$$

As G is generated by generalized reflections, [4], Theorem 4, yields the first equality in

$$\mathfrak{D}_{R|P} = \prod_{h \in \mathcal{P}} i[h] = \prod_{i=1}^s \prod_{G[h] \in \mathcal{G}_i} \prod_{\substack{g \in G[h] \\ g \neq \text{id}}} i[g] = \prod_{i=1}^s \prod_{i[h] \in \mathcal{I}_i} i[h]^{n_i-1} = \prod_{i=1}^s (R x_i)^{n_i-1} = R z.$$

On the other hand, Theorem 2 shows that P is the fixed subring of Q under the action of the group $G_{\text{ab}} = G/G'$, whose generators $\rho(h_1), \dots, \rho(h_s)$ are generalized reflections with inertia ideals in Q generated by x_1, \dots, x_s , respectively. The set \mathcal{P}' of generalized

reflections in G_{ab} is the disjoint union $\bigsqcup_{i=1}^s (\rho(G_i) \setminus 1)$. Thus, denoting by $i[h']$ the inertia ideal in Q of $h' \in \mathcal{P}'$, by another application of [4], Theorem 4, we obtain

$$\mathfrak{D}_{Q|P} = \prod_{h' \in \mathcal{P}'} i[h'] = \prod_{i=1}^s \prod_{\substack{h' \in \rho(G_i) \\ h' \neq \text{id}}} i[h'] = \prod_{i=1}^s i[\rho(h_i)]^{n_i-1} = \prod_{i=1}^s (Q x_i)^{n_i-1} = Q z.$$

Given a fractional ideal \mathfrak{a} of R , we denote by $\bar{\mathfrak{a}}$ its divisorial hull ($R : (R : \mathfrak{a})$), where colons are computed in the fraction field R_0 . We then have equalities

$$Rz = \mathfrak{D}_{R|P} = \overline{\mathfrak{D}_{R|Q} \mathfrak{D}_{Q|P}} = \overline{z \mathfrak{D}_{R|Q}} = z \overline{\mathfrak{D}_{R|Q}} = z \mathfrak{D}_{R|Q}$$

where the first and third come from the computations above, the second is the transitivity of the different, cf. [5], Lemma 3.10.1, the fourth follows from the definition, and the last is due to the divisoriality of the different. The result is $\mathfrak{D}_{R|Q} = R$, as desired. \square

4. FACTORIAL EXTENSIONS

We start with generically Galois factorial extensions of regular local rings. The first two assertions of the next result are already contained in [8], Theorem 2.1. The remaining ones give a rather more general and somewhat more precise version of [8], Theorem 2.4.

Theorem 5. *Let (R, \mathfrak{m}) be a local factorial domain which is the integral closure of a regular local ring P' in a Galois extension of the fraction field P'_0 . The following then hold.*

The group $G = \{g \in \text{Gal}(R_0|P'_0) \mid g(a) - a \in \mathfrak{m} \text{ for all } a \in R\}$ of automorphisms of R is generated by generalized reflections, the ring $P = R^G$ is regular, and P is étale over P' .

If $|G|$ is invertible in P , then R is a complete intersection.

If furthermore R has a primitive root of unity of order equal to the exponent of G_{ab} , and Q is the integral closure of P in the largest abelian extension of P_0 in R_0 , then there exist $y_1, \dots, y_s \in P$ such that Q is a complete intersection $P[Y_1, \dots, Y_s]/(Y_1^{n_1} - y_1, \dots, Y_s^{n_s} - y_s)$ and the extension $Q \hookrightarrow R$ is divisorially unramified.

Proof. It is classical that P is étale over P' , cf. [13], Théorème X.1. This implies that P is regular. It is shown in the proof of [15], Théorème 2', that a finite group G which acts on a factorial local ring with a regular ring of invariants is generated by generalized reflections. If $|G|$ is invertible, then R is a complete intersection by [4], Corollary 3.(i). The remaining assertions have been gleaned from Theorems 2 and 4 above. \square

To relax the condition on the fraction field extension in Theorem 5, we focus on two field extensions associated with a finite field extension $K \hookrightarrow K'$. The (quasi-)Galois closure of K in K' is the unique largest (quasi-)Galois extension of K contained in K' ; a (quasi-)Galois shell of K' over K is the unique up to isomorphism smallest quasi-Galois extension of K which contains K' .

We say that an extension S of P is *normally divisorially ramified* over P if there exists a normal ring $D \subseteq S$, such that P is the fixed ring of a finite group of automorphisms of D , and S is divisorially unramified over D .

Theorem 6. *Consider a finite residually trivial extension of a regular local ring P by a normal local domain E , such that the fraction field extension $P_0 \hookrightarrow E_0$ is Galois, $n = \text{rank}_P E$ is invertible in P , and E_0 has a primitive n 'th root of unity.*

Let S be a ring between P and E , let R be the integral closure of P in the Galois closure of P_0 in S_0 , and let T be the integral closure of P in a Galois shell of S_0 over P_0 .

If S is factorial and normally divisorially ramified over P , then the extensions $R \hookrightarrow S \hookrightarrow T$ are divisorially unramified and the ring R is factorial (so Theorem 5 applies to R).

The integral closure of a henselian domain P in a quasi-Galois shell of a finite field extension of P_0 is local, finite, and residually trivial over P , so we have

Corollary 7. *If P is a henselian (e.g., complete) regular local ring with algebraically closed residue field of characteristic zero, $P \hookrightarrow S$ is a finite normally divisorially ramified extension, and S is factorial, then the conclusions of the theorem hold. \square*

Remark. The corollary is closely related to [9], Theorem 4.2, which works from the hypothesis that the divisorial ramification is *uniform* in the sense that all height one primes of S lying over the same prime of P have equal ramification indices (an additional condition that S has an isolated singularity is never used in the proof). The assumption of ‘uniformity’ is applied through [9], Theorem 1.5, but there is a gap in the proof of that result: the equality starting on its second line holds only after multiplication by some $u \in S^\times$, so the balance of the argument requires the existence in S of an appropriate root of u .

Such a root exists when P is complete with algebraically closed residue field of characteristic zero. By [9], Theorem 1.5, if S is factorial, P is regular, and the extension is uniformly ramified in codimension one, then it is normally divisorially ramified.

We precede the proof of the theorem by a variation on [9], Proposition 1.2.

Lemma 8. *Let $P \hookrightarrow D \hookrightarrow S$ be finite extensions of normal domains such that D is generically Galois over P , and let $P_0 \hookrightarrow L$ be a quasi-Galois shell of S_0 over P_0 . If S is divisorially unramified over D , then so is the integral closure T of P in L .*

Proof. As S is divisorially unramified over D , its field of fractions S_0 is separable over D_0 , so S_0 is separable over P_0 , hence L is Galois over P_0 . Note that T is the integral closure in L of the composite of the rings $\{g(S) \mid g \in \text{Gal}(L|P_0)\}$. Each $g(S)$ is normal, finite, and divisorially unramified over $g(D)$. As $g(D) = D$ for all g (because $P_0 \hookrightarrow D_0$ is Galois), it suffices to show that if S_1 and S_2 are normal domains which are finite and divisorially unramified extensions of D , then so is the integral closure $(S_1 S_2)'$ of their composite $S_1 S_2$.

Let $\mathfrak{p} \subset D$ be a prime of height one. The semi-local Dedekind domains $(S_1)_{\mathfrak{p}}$ and $(S_2)_{\mathfrak{p}}$ are unramified over $D_{\mathfrak{p}}$, hence by [3], Proposition A.1, the extension $D_{\mathfrak{p}} \hookrightarrow (S_1)_{\mathfrak{p}}(S_2)_{\mathfrak{p}}$ is unramified. As $D_{\mathfrak{p}}$ is regular, so is $(S_1)_{\mathfrak{p}}(S_2)_{\mathfrak{p}} = (S_1 S_2)_{\mathfrak{p}}$. In particular, $(S_1 S_2)_{\mathfrak{p}}$ is integrally closed. Since integral closure commutes with localization, we have $(S_1 S_2)_{\mathfrak{p}} = (S_1 S_2)'_{\mathfrak{p}}$. It follows that $(S_1 S_2)'$ is divisorially unramified over D . \square

The proof of Theorem 6 draws upon that of [9], Theorem 4.2; the preceding results allow for an extension of the scope of that argument, and for significant simplifications.

Proof of Theorem 6. By assumption, there is a normal domain D such that $P \hookrightarrow D \hookrightarrow S$, the fraction field extension $P_0 \hookrightarrow D_0$ is Galois, and S is divisorially unramified over D . Lemma 8 shows T is divisorially unramified over D . We set $G = \text{Gal}(T_0|P_0)$, and note that $S = T^H$, where $H = \text{Gal}(T_0|S_0) \leq G$. If N is the the smallest normal subgroup of G which contains H , then T_0^N is the largest Galois extension of P_0 contained in S_0 . The integral closure R of P in T_0^N is equal to T^N , and R necessarily contains D . As $D \hookrightarrow T$ is divisorially unramified, by [3], Proposition A.2, so are the intermediate extensions $R \hookrightarrow S \hookrightarrow T$.

Next we analyze the homomorphisms of abelian groups in the diagram

$$\begin{array}{ccccc} \text{Cl}(S) & \xrightarrow[\cong]{\alpha} & \text{H}^1(H, T^\times) & \xrightarrow[\cong]{\beta} & \text{Hom}(H, k^\times) \\ & & & & \uparrow \gamma \\ \text{Cl}(R) & \xrightarrow[\alpha']{\cong} & \text{H}^1(N, T^\times) & \xrightarrow{\beta'} & \text{Hom}(N, k^\times) \end{array}$$

where $\text{Cl}(\)$ denotes divisor class groups and k stands for the common residue field of all the local rings under consideration. The maps α and α' are given by Galois descent; they are bijective because T is divisorially unramified over S and over R , cf. [14], Theorem III.1.1, or [7], Theorem 16.1. The map β' is injective by Lemma 3. The roots of unity in E_0 live in its integrally closed subring E , so the same lemma proves β is bijective.

The isomorphisms in the top row of the diagram and the factoriality of S yield $\text{Hom}(H, k^\times) = 0$. Assuming $\text{Hom}(N, k^\times) \neq 0$, let $\xi: N \rightarrow k^\times$ be a nontrivial group homomorphism. As N is the subgroup of G generated by all the conjugates of H , there exists $g \in G$ such that $\xi(gHg^{-1}) \neq 1$. But then the composition of ξ with the isomorphism $H \rightarrow gHg^{-1}$ given by conjugation provides a nontrivial homomorphism $H \rightarrow k^\times$, which contradicts the already established triviality of $\text{Hom}(H, k^\times)$. Thus, $\text{Hom}(N, k^\times) = 0$. The injectivity of the maps in the bottom row implies $\text{Cl}(R) = 0$, hence R is factorial. \square

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