HOMOLOGICAL DIMENSIONS AND RELATED INVARIANTS OF MODULES OVER LOCAL RINGS

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Explanation

This paper evolved out of two lectures given at the Workshop of ICRA IX[1].

In the lectures I tried to present to a group of experts in representation theory of Artin algebras some recent developments in commutative algebra. To bridge the gap between subjects, I looked for theorems that could be appreciated directly off the blackboard in mathematically meaningful detail. The search ended close to home: many results on free resolutions over commutative local rings can be recast as statements on projective resolutions over artinian rings (that just happen to be commutative). A further slate of assertions becomes intelligible when some property of local rings (e.g. 'Gorenstein') is replaced by an adequate condition on artinian algebras (e.g. 'quasi-Frobenius'). Thus, I chose to discuss with a not-necessarily-commutative audience problems whose solutions may be influenced by a commutativity hypothesis, but whose statements do not depend on one.

Community of interest and hand waving provided some coherence to the oral exposition. Work on the article rapidly convinced me that the written version needs a more robust structure. Trying to find one I deleted material, expanded a few themes, and embarked upon a series of rewritings. The text that emerged focuses on measures of resolutions such as length, size, or width, and on the ways that their values affect the homological life of modules.

I want to thank the organizers for an opportunity to present some favorite topics, the participants for their willingness to cross mathematical frontiers, and the editors for trusting, against all odds, that this paper will be completed.

Introduction

We discuss ramifications and extensions of the classical notions of projective dimension and injective dimension of modules.

Various hypotheses are made on the base ring at different places in the text. It is essentially arbitrary in Sections 2 and 4, noetherian in Section 3, and graded in Section 10. Everywhere else, ring is assumed to be *local*, which here is taken to mean 'commutative noetherian ring with unique maximal ideal', and modules are assumed to be *finite*, that is, finitely generated. For the convenience of some readers, in Appendix B we recall the definitions of a few basic notions of local algebra, and provide background.

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[1] For the uninitiated: ICRA stands for International Conference on Representation of Algebras; the ninth one was held in Beijing from August 20 to September 1, 2000.

Modules of finite projective dimension or of finite injective dimension over a local ring have rather special properties. Some highlights are recalled in Section 1 with a dual purpose in mind: to provide a historical perspective and to indicate the kind of results to be expected (or desired) for modules of finite generalized homological dimension. An important area of research in commutative algebra—the structure theory of finite free resolutions—is omitted because its results heavily depend on determinantal ideals, and so have no clear counterparts outside of the commutative context.

Early generalizations of the classical dimensions measured the minimal length of resolutions (respectively, coresolutions) constructed using modules from a specific class. If such a class does not consist of projective (respectively, injective) modules, then for satisfactory results one needs extra conditions either on the maps or on the modules used in the (co)resolutions. In Section 2 we review well known solutions to the problems that arise. Our conventions for handling complexes and resolutions are explained in Appendix A.

The best studied non-classical homological dimension is the Gorenstein dimension, which is the subject of Section 3. The not quite traditional approach there is based on explicit constructions of complexes, and has applications to computations of relative cohomology and Tate cohomology.

It is classically known that each module has a 'minimal' injective coresolutions, and that some modules—in particular, finite ones over local rings—also have minimal projective resolutions. In either case the corresponding complex is unique up to non-unique isomorphism, and splits off every (co)resolution as a direct summand with contractible direct complement. Minimal (co)resolutions are discussed in Section 4. We use a new approach, based on a notion of minimality applicable to arbitrary complexes. It allows one to define and study minimal resolutions that do not necessarily consist of projectives.

Over the last two decades, new invariants of modules have been introduced by taking into account the 'size' of modules appearing in resolutions or coresolutions. It is natural to measure a free module by its rank; the Betti numbers of a module are the ranks of the free modules in its minimal free resolution. A measure for the size of an injective module is defined by using the structure theorem for injective module over commutative noetherian rings; it leads to the notion of Bass numbers. In Section 5 we discuss bounds on sequences of Betti numbers and of Bass numbers.

In all cases when it is known, the asymptotic behavior of these sequences belongs to one of two types—polynomial or exponential. Sample results pertaining to these types of growth are discussed in Sections 6 and 9, respectively. A features of many of their proofs is the use of subtle analogies with 'distant' branches of mathematics, such as group cohomology or rational homotopy theory. The invariant used to measure the growth of resolutions on a polynomial scale is known as the complexity of the module, the one used to gauge growth on the exponential scale has been called its curvature. Besides providing classifying parameters in situations where other dimensions are infinite, the values of these asymptotic invariants impact the vanishing of (co)homology.

In Section 8 certain kinds of resolutions are paired off with specified changes of the base ring, so as to produce a variety of new homological dimensions. The rationale comes from the fundamental hierarchy of properties of local rings. Section 7 focuses on complete intersection dimension. When it is finite, precise results are available on the asymptotic behavior of free resolutions.

A graded module over a graded ring has graded Betti numbers, which can be used to measure the 'width' of its minimal free resolution. This new parameter leads to the notion of Castelnuovo-Mumford regularity of a module. It is the subject of the final Section 10, where we show that in the commutative case regularity behaves in some respects like another homological dimension.

Throughout the text, and to the best of my knowledge, I have tried to give references to the original publications. The intent was to provide readers from a different area with a sense of the dynamics of the developments reported below. On the other hand, systematic exposition in research monographs, textbooks, or survey articles often offer the best introduction to a field. At the end of the relevant sections I have collected references to such sources.

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1. Classical dimensions

In this section $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module.

A hypothesis that a classical (that is, projective or injective) dimension of M is finite imposes stringent restrictions on the structure of M. We illustrate this fact and give a flavor of things to come by reviewing basic results.

1.1. Finite projective dimension. When a projective dimension is finite, its value is determined by a famous equality.

1.1.1. Theorem (Auslander-Buchsbaum [5]). *If* $\operatorname{proj\,dim}_R M$ *is finite, then*

$$\operatorname{proj\,dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$$

All admissible values of $\operatorname{proj\,dim}_R M$ are indeed achieved.

1.1.2. Example. Set $d = \operatorname{depth} R$, choose an R-regular sequence $a_1, \dots, a_d$, and set $\boldsymbol{y}_i = a_1, \dots, a_i$ for $i = 0, \dots, d$. By Theorem (B.1.2), the Koszul complex $\mathrm{K}(\boldsymbol{y}_i; R)$ is exact, hence is a free resolution of the R-module $N_i = R/(\boldsymbol{y}_i)$. This resolution is minimal, so $\operatorname{proj\,dim}_R(N_i) = i$, cf. (4.2).

1.2. Regular local rings. One of the most influential results in commutative algebra describes the local rings of finite global dimension as being the regular rings. It is a formal consequence of the following more precise statement.

1.2.1. Theorem (Auslander-Buchsbaum [4], [5]; Serre [64]). *The following conditions are equivalent.*

(i) *R is regular.*
(ii) $\operatorname{proj\,dim}_R N \le \operatorname{depth} R$ *for every finite R-module N.*
(iii) $\operatorname{proj\,dim}_R k < \infty$.

The implication (i) $\implies$ (ii) may be viewed as the ultimate generalization of Hilbert's Syzygy Theorem, cf. also (10.2.2). The implication (iii) $\implies$ (i) is considerably strengthened below (note that $\operatorname{proj\,dim}_R 0 = -\infty$, cf. (2.2.3)).

1.2.2. Theorem (Levin-Vasconcelos [49]). *If there exists a finite module N such that* $\operatorname{proj\,dim}_R(\mathfrak{m}^i N)$ *is finite for some $i \ge 1$, then R is regular.*

Our discussion of injective dimension parallels that of projective dimension.

1.3. Finite injective dimension. The first result may surprise.

1.3.1. Theorem (Bass [22]). *If* $\operatorname{inj\,dim}_R M$ *is finite, then*

$$\operatorname{inj\,dim}_R M = \operatorname{depth} R$$

Some examples of modules of finite injective dimension are easy to produce.

1.3.2. Example. Every Cohen-Macaulay ring R has finite modules of finite injective dimension. Indeed, choose an R-regular sequence $\boldsymbol{y}$ of length $d = \operatorname{depth} R$. By Example (1.1.2), the module $N_d = R/(\boldsymbol{y})$ has a finite free resolution $\boldsymbol{K}$. If E is an injective envelope of k over R, then the complex $\operatorname{Hom}_R(\boldsymbol{K}, E)$ is a finite injective coresolution of $N = \operatorname{Hom}_R(N_d, E)$. Furthermore, $\operatorname{length}_R(N) = \operatorname{length}_R(N_d) < \infty$, with equality obtained by induction using the exactness of the functor $\operatorname{Hom}_R(-, E)$, and inequality due to the fact that the ring R is Cohen-Macaulay.

In 1963, Bass [22] asked if finite modules of finite injective dimension exist only over Cohen-Macaulay rings. It took 25 years to get a complete answer.

1.3.3. Theorem (Peskine-Szpiro [56], Hochster [47], Roberts [60]). *If the module* M *has finite injective dimension, then the ring* R *is Cohen-Macaulay.*

We go on to homological characterizations of the Gorenstein property; for descriptions in terms of other homological dimensions cf. Theorem (3.5.1).

1.4. Gorenstein local rings. The classical description runs as follows:

1.4.1. Theorem (Bass [22]). R *is Gorenstein if and only if* $\operatorname{inj\,dim}_R R < \infty$.

This characterization is contained in the more precise result below.

1.4.2. Theorem (Foxby [40]). *The following conditions are equivalent.*

(i) R *is Gorenstein.*
(ii) *Every finite module* N *with* $\operatorname{proj\,dim}_R N < \infty$ *has* $\operatorname{inj\,dim}_R N < \infty$.
(iii) *There exists a finite module* N *such that* $\operatorname{proj\,dim}_R N$ *and* $\operatorname{inj\,dim}_R N$ *are both finite.*

Expositions. The monographs of Hochster [47], Northcott [54], Roberts [59], Evans and Griffith [38], Strooker [66], Roberts [61] deal extensively (some—exclusively) with modules of finite projective dimension or finite injective dimension over commutative noetherian rings. The textbooks of Matsumura [52], Bruns and Herzog [26], Eisenbud [35] contain many relevant results.

2. Dimensions from resolutions

Let R be an associative ring, and let $\mathcal{M} = \mathcal{M}(R)$ be the category of R-modules. We fix an abelian subcategory $\mathcal{F} \subseteq \mathcal{M}$, a subcategory $\mathcal{C} \subseteq \mathcal{F}$, and let $\mathcal{P} \subseteq \mathcal{F}$ denote the subcategory whose objects are the projective R-modules in $\mathcal{F}$, cf. (A.1) for some general conventions on categories of modules.

Two standard procedures for replacing projective resolutions by more general ones lead to homological dimensions measuring rather different effects.

2.1. Relative dimension. Under this approach, modules other than projectives are allowed in resolutions, but not all homomorphisms can be used as differentials. The goal is to ensure the existence of comparison morphisms for resolutions, leading to a theory of relative cohomology functors. The payoff is that whenever a module admits a resolution, its homological dimension can be computed from the vanishing of an appropriate cohomological functor. The downside is that the class of modules of finite homological dimension may shrink, due to insufficiency of proper maps.

2.1.1. Proper resolutions. A sequence $\boldsymbol{E}$ of homomorphisms in $\mathcal{F}$ is *proper exact* (with respect to $\mathcal{C}$) if the sequence $\operatorname{Hom}_R(C, \boldsymbol{E})$ is exact for all $C \in \mathcal{C}$. A $\mathcal{C}$-resolution $\gamma\colon \boldsymbol{C} \to M$ is *proper* if it yields a proper exact sequence

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\gamma_0} M \longrightarrow 0$$

When $\mathcal{C} = \mathcal{P}$ we are in the *absolute case*, where every $\mathcal{C}$-resolution is proper.

2.1.2. Relative cohomology. Let $\overline{\mathcal{C}}$ denote the subcategory of $\mathcal{F}$ whose objects are the R-modules admitting some proper $\mathcal{C}$-resolution.

To construct cohomology functors one uses a *Comparison Lemma* for proper resolutions, proved by the usual arguments for the projective case.

2.1.3. Lemma. *If $\epsilon\colon \boldsymbol{C} \to M$ and $\epsilon'\colon \boldsymbol{C}' \to M'$ are proper $\mathcal{C}$-resolutions, then for every R-linear homomorphism $\mu\colon M \to M'$ there is a unique up to homotopy* comparison morphism $\widetilde{\mu}\colon \boldsymbol{C} \to \boldsymbol{C}'$ *that that satisfies $\epsilon'\widetilde{\mu} = \mu\epsilon$.*

Choosing for each $M \in \overline{\mathcal{C}}$ a proper resolution $\boldsymbol{C} \to M$, define for each $n \in \mathbb{Z}$ and each $N \in \mathcal{M}(R)$ a *relative cohomology group*

$$\operatorname{Ext}^n_{\mathcal{C}}(M, N) = \operatorname{H}^n \operatorname{Hom}_R(\boldsymbol{C}, N)$$

By the Comparison Lemma, $(M, N) \mapsto \operatorname{Ext}^n_{\mathcal{C}}(M, N)$ yields a functor

$$\operatorname{Ext}^n_{\mathcal{C}} : \overline{\mathcal{C}}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\mathbb{Z})$$

Choosing for each $M \in \overline{\mathcal{C}}$ a projective resolution $\pi\colon \boldsymbol{P} \to M$ and a morphism of complexes $\phi\colon \boldsymbol{P} \to \boldsymbol{C}$ lifting id_M one obtains homomorphisms

$$\varepsilon^n_{\mathcal{C}}(M, N) = \operatorname{H}^n \operatorname{Hom}_R(\phi, N)\colon \operatorname{Ext}^n_{\mathcal{C}}(M, N) \longrightarrow \operatorname{Ext}^n_R(M, N)$$

These maps produce morphisms of functors $\varepsilon^n_{\mathcal{C}}\colon \operatorname{Ext}^n_{\mathcal{C}} \to \operatorname{Ext}^n_R$ which commute with connecting homomorphisms. Furthermore, $\varepsilon^0_{\mathcal{C}}\colon \operatorname{Ext}^0_{\mathcal{C}} \to \operatorname{Hom}_R$ is an isomorphism and $\varepsilon^1_{\mathcal{C}}\colon \operatorname{Ext}^1_{\mathcal{C}} \to \operatorname{Ext}^1_R$ is a monomorphism.

Relative cohomology looks like a cohomological functor: every *proper* exact sequence in either argument induces a natural cohomology exact sequence.

2.1.4. Yoneda classes. For all $M, N \in \mathcal{F}$, Yoneda congruence relations involving *only* proper short exact sequences define equivalence classes of proper exact sequences of length n, starting at N and ending at M. The result is an *abelian group of proper extensions* that depends functorially on M and N. For every $M \in \overline{\mathcal{C}}$ this group is canonically isomorphic with $\operatorname{Ext}^n_{\mathcal{C}}(M, N)$. The map sending the class of each proper sequence to its congruence class with respect to Yoneda equivalence using *all* short exact sequences is a morphism of extension functors. For $M \in \overline{\mathcal{C}}$, the canonical isomorphism above transforms this morphism into $\varepsilon^n\colon \operatorname{Ext}^n_{\mathcal{C}} \to \operatorname{Ext}^n_R$.

Using properties of relative Ext functors outlined above, one gets the next result by repeating *verbatim* the classical arguments for the absolute case.

2.1.5. Proposition-Definition. *Let $\mathcal{C}$ be a subcategory of $\mathcal{F}$.*

For every $M \in \overline{\mathcal{C}}$ and for each $g \in \mathbb{N}$ the following are equivalent.

(i) *There exists a proper $\mathcal{C}$-resolution $\boldsymbol{C} \to M$ with $C_n = 0$ for all $n > g$.*
(ii) *Every proper $\mathcal{C}$-resolution $\boldsymbol{C} \to M$ has $\Omega^g \boldsymbol{C} \in \mathcal{C}$.*
(iii) $\operatorname{Ext}^{g+1}_{\mathcal{C}}(M, -) = 0$.
(iv) $\operatorname{Ext}^n_{\mathcal{C}}(M, -) = 0$ *for all $n > g$.*

When they hold M has relative dimension *at most g with respect to $\mathcal{C}$, denoted*

$$\operatorname{rel\,dim}_{\mathcal{C}} M \leq g$$

For $M \in \mathcal{F} \smallsetminus \overline{\mathcal{C}}$ one sets $\operatorname{rel\,dim}_{\mathcal{C}} M = \infty$.

2.2. Resolving dimension. In this approach no restrictions are placed on maps in resolutions. Thus, enlarging the class of modules allowed in resolutions one gets a corresponding enlargements of the class of modules of finite homological dimension. Here the caveat is: for a dimension to be computable from *every* resolution, the class of allowable modules cannot be arbitrary.

2.2.1. Resolving subcategories. A subcategory $\mathcal{C} \subseteq \mathcal{F}$ is *resolving* for $\mathcal{F}$ if $\mathcal{C}$ is closed under extensions and under kernels of epimorphisms, and contains enough projectives for $\mathcal{F}$, in the sense that every $M \in \mathcal{F}$ is the homomorphic image of some module from $\mathcal{C} \cap \mathcal{P}$. *Every* module $M \in \mathcal{F}$ has a $\mathcal{C}$-resolution.

2.2.2. Proposition-Definition (Auslander-Bridger [3]). *Let $\mathcal{C}$ be a resolving subcategory for $\mathcal{F}$.*

For every $M \in \mathcal{M}$ and for each $g \in \mathbb{N}$ the following are equivalent.

(i) *There exists a $\mathcal{C}$-resolution $\boldsymbol{C} \to M$ with $C_n = 0$ for $n > g$.*
(ii) *Every $\mathcal{C}$-resolution $\boldsymbol{C} \to M$ has $\Omega^g \boldsymbol{C} \in \mathcal{C}$.*
(iii) *There exists a $\mathcal{P}$-resolution $\boldsymbol{P} \to M$ with $\Omega^g \boldsymbol{P} \in \mathcal{C}$.*

When they hold, M is said to have $\mathcal{C}$-dimension *at most g, denoted*

$$\mathcal{C}\text{-dim}\, M \leq g$$

The next observations follow directly from the definitions.

2.2.3. Remark. If $\mathcal{C}$ is resolving for $\mathcal{F}$, then for each $M \in \mathcal{F}$ one has:

$$\begin{aligned} \mathcal{C}\text{-dim}\, M = -\infty &\iff \operatorname{rel\,dim}_{\mathcal{C}} M = -\infty \iff M = 0 \\ \mathcal{C}\text{-dim}\, M = 0 \quad &\iff \operatorname{rel\,dim}_{\mathcal{C}} M = 0 \quad \iff M \in \mathcal{C} \smallsetminus 0 \\ \mathcal{C}\text{-dim}\, M &\leq \min\{\operatorname{rel\,dim}_{\mathcal{C}} M\,,\, \operatorname{proj\,dim}_R M\} \end{aligned}$$

Standard use of mapping cones and the Horseshoe Lemma lead to

2.2.4. Proposition. *For every exact sequence $0 \to M \to M' \to M'' \to 0$ in $\mathcal{F}$ there is an inequality*

$$\mathcal{C}\text{-dim}\, M \leq \max\{\mathcal{C}\text{-dim}\, M', \mathcal{C}\text{-dim}\, M'' - 1\}$$

If $\mathcal{C}\text{-dim}\, M' \neq \mathcal{C}\text{-dim}\, M''$, then equality holds.

2.2.5. Example. The subcategory $\mathcal{P}$ is resolving for $\mathcal{F}$ if and only if $\mathcal{P}$ has enough projectives for $\mathcal{F}$; in that case $\mathcal{P}\text{-dim}\, M = \operatorname{proj\,dim}_R M$ for $M \in \mathcal{F}$.

Expositions. In discussing relative homological algebra we have followed the memoir of Eilenberg and Moore [33] and the book of MacLane [50].

3. Gorenstein dimension

In this section R is a noetherian (on both sides) ring, and M a finite R-module. We let $\mathcal{F} \subseteq \mathcal{M}$ denote the subcategory of finite R-modules, and $\mathcal{P} \subseteq \mathcal{F}$ the subcategory of projective R-modules.

3.1. Gorenstein dimension. An R-module G is *reflexive* if it is finite and

$$G \cong G^{**}$$

This happens if and only if the *canonical map* $G \to G^{**}$ is bijective, cf. (A.1).

We say that G is *totally reflexive* if it is reflexive and satisfies

$$\operatorname{Ext}_R^n(G,R) = 0 = \operatorname{Ext}_{R^\circ}^n(G^*,R) \quad \text{for all} \quad n \geq 1$$

These modules were introduced in [3] as *modules of G-dimension zero.*

Let $\mathcal{G} \subseteq \mathcal{F}$ be the subcategory of totally reflexive modules.

3.1.1. Lemma. *The subcategory $\mathcal{G}$ is resolving for $\mathcal{F}$.*

By Proposition (2.2.2) the number $\mathcal{G}\text{-}\dim M$ defined in (2.2.1) can be computed from any $\mathcal{G}$-resolution. This invariant, introduced by Auslander and Bridger [3], is called the *Gorenstein dimension* or *G-dimension* of M; the name is explained by Theorem (3.5.1). To use standard notation, we set

$$\operatorname{G-dim}_R M = \mathcal{G}\text{-}\dim M$$

In the next result the first inequality is obtained by an easy induction on $\operatorname{G-dim}_R M$, the second from Proposition (2.2.2). The conditions for equality establish the precise sense in which G-dimension refines projective dimension.

3.1.2. Theorem (Auslander-Bridger [3])**.** *There are inequalities*

$$\sup\{n \in \mathbb{N} \mid \operatorname{Ext}_R^n(M,R) \neq 0\} \leq \operatorname{G-dim}_R M \leq \operatorname{proj\,dim}_R M$$

and equalities hold to the left of any finite dimension.

The equivalence (i) $\iff$ (ii) below is proved in [3] over local rings, and in [19] in general. The implication (ii) $\implies$ (iii) is due to Zaks [69].

3.1.3. Theorem. *For each integer $d \geq 0$ the following are equivalent.*

(i) *Every finite module, left or right, has G-dimension at most d.*
(ii) $\operatorname{inj\,dim}_R R \leq d$ *and* $\operatorname{inj\,dim}_{R^\circ} R \leq d$.
(iii) $\operatorname{inj\,dim}_R R = \operatorname{inj\,dim}_{R^\circ} R \leq d$.

3.2. Relative dimension. Let $\operatorname{rel\,dim}_{\mathcal{G}} M$ be the relative dimension of M with respect to $\mathcal{G}$, as in (2.1). To relate it to G-dimension, we say that a $\mathcal{G}$-resolution $\boldsymbol{G} \to M$ is *strict* if G_n is projective for all $n \geq 1$.

3.2.1. Theorem. *For each integer $g \geq 0$ the following are equivalent.*

(i) $\operatorname{G-dim}_R M \leq g$, *that is, M has a $\mathcal{G}$-resolution of length $\leq g$.*
(ii) $\operatorname{rel\,dim}_{\mathcal{G}} M \leq g$, *that is, M has a proper $\mathcal{G}$-resolution of length $\leq g$.*
(iii) *M has a strict $\mathcal{G}$-resolution of length $\leq g$.*

In particular, $\operatorname{G-dim}_R M = \operatorname{rel\,dim}_{\mathcal{G}} M$.

The implication (ii) $\implies$ (i) is trivial, while (i) $\implies$ (iii) is shown by an explicit construction. Finally, (iii) $\implies$ (ii) holds because strict resolutions are proper, by the next lemma, easily proved by induction on $\operatorname{proj\,dim}_R N$.

3.2.2. Lemma. *If G is a totally reflexive R-module, then* $\operatorname{Ext}_R^n(G,N) = 0$ *for every R-module N of finite projective dimension and all $n \geq 1$.* □

3.3. Tate cohomology. A *totally acyclic* complex is a complex $\boldsymbol{T}$ such that T_n is finite projective for each $n \in \mathbb{Z}$ and the following condition holds

$$\mathrm{H}_n(\boldsymbol{T}) = 0 = \mathrm{H}_n(\boldsymbol{T}^*) \quad \text{for all} \quad n \in \mathbb{Z}$$

A *complete resolution* of M is a diagram $\boldsymbol{T} \xrightarrow{\vartheta} \boldsymbol{P} \xrightarrow{\pi} M$ with a $\mathcal{P}$-resolution π, a totally acyclic complex $\boldsymbol{T}$, a morphism ϑ with ϑ_n bijective for $n \gg 0$.

3.3.1. Theorem. *For each integer $g \geq 0$ the following are equivalent.*

(i) $\text{G-dim}_R M \leq g$.
(ii) *M has a complete resolution $\vartheta \colon \boldsymbol{T} \to \boldsymbol{P}$ with ϑ_n bijective for all $n \geq g$.*
(iii) *M has a complete resolution $\vartheta \colon \boldsymbol{T} \to \boldsymbol{P}$ with ϑ_n bijective for all $n \geq g$ and surjective for all $n \in \mathbb{Z}$*

The proof yields explicit constructions of the desired complexes.

To define cohomology functors on the basis of complete resolutions, one needs a corresponding *Comparison Lemma*; the one below is from [19].

3.3.2. Lemma. *If $\boldsymbol{T} \xrightarrow{\vartheta} \boldsymbol{P} \xrightarrow{\pi} M$ and $\boldsymbol{T}' \xrightarrow{\vartheta'} \boldsymbol{P}' \xrightarrow{\pi'} M'$ are complete resolutions, then for each R-linear map $\mu \colon M \to M'$ there exists a unique up to homotopy morphism $\overline{\mu}$, making the right hand square of the diagram*

$$\begin{array}{ccccc}
\boldsymbol{T} & \xrightarrow{\vartheta} & \boldsymbol{P} & \xrightarrow{\pi} & M \\
\downarrow{\scriptstyle \widehat{\mu}} & & \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\
\boldsymbol{T}' & \xrightarrow{\vartheta'} & \boldsymbol{P}' & \xrightarrow{\pi'} & M'
\end{array}$$

commute, and for each choice of $\overline{\mu}$ there exists a unique up to homotopy morphism $\widehat{\mu}$, making the left hand square commute up to homotopy.

Let $\widetilde{\mathcal{G}}$ be the subcategory of $\mathcal{F}$ whose objects are the modules of finite G-dimension. Choosing for each $M \in \widetilde{\mathcal{G}}$ a complete resolution $\boldsymbol{T} \xrightarrow{\vartheta} \boldsymbol{P} \xrightarrow{\pi} M$, one defines for every $N \in \mathcal{M}$ and every $n \in \mathbb{Z}$ a *Tate cohomology group*

$$\widehat{\mathrm{Ext}}^n_R(M, N) = \mathrm{H}^n \operatorname{Hom}_R(\boldsymbol{T}, N)$$

By the Comparison Lemma, $(M, N) \mapsto \widehat{\mathrm{Ext}}^n_R(M, N)$ yields a functor

$$\widehat{\mathrm{Ext}}^n_R \colon \widetilde{\mathcal{G}}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\mathbb{Z})$$

for every $n \in \mathbb{Z}$, and the homomorphisms

$$\varepsilon^n_R(M, N) = \mathrm{H}^n \operatorname{Hom}_R(\vartheta, N) \colon \operatorname{Ext}^n_R(M, N) \to \widehat{\mathrm{Ext}}^n_R(M, N)$$

yield morphisms of functors $\varepsilon^n_R \colon \operatorname{Ext}^n_R \to \widehat{\mathrm{Ext}}^n_R$, invertible for $n > \text{G-dim}_R M$. Clearly, $\widehat{\mathrm{Ext}}^n_R(M, -)$ is a cohomological functor, each short exact sequence of modules in $\widetilde{\mathcal{G}}$ induces a natural exact sequence of Tate cohomology groups, and ε^n_R defines morphisms of the corresponding long exact sequences.

The prototype of Tate cohomology is precisely what the name suggests:

3.3.3. Example. Let Π be a finite group, $R = \mathbb{Z}[\Pi]$ its group ring, and $\mathbb{Z}$ the R-module with trivial action: $gm = m$ for all $g \in \Pi$ and $m \in \mathbb{Z}$.

Choose a resolution $\epsilon \colon \boldsymbol{P} \to \mathbb{Z}$ by finite free R-modules. For each i, the action $(g\alpha)(p) = \alpha(g^{-1}p)$ turns $\operatorname{Hom}_{\mathbb{Z}}(P_i, \mathbb{Z})$ into a free left Π-module. Furthermore, $\mathrm{H}_0(\operatorname{Hom}_{\mathbb{Z}}(\boldsymbol{P}, \mathbb{Z})) = \mathbb{Z}$, and $\mathrm{H}_i(\operatorname{Hom}_{\mathbb{Z}}(\boldsymbol{P}, \mathbb{Z})) = 0$ for $i \neq 0$. Splice $\boldsymbol{P}$ with the complex $\operatorname{Hom}_{\mathbb{Z}}(\boldsymbol{P}, \mathbb{Z})$, shifted one degree to the right, along the R-linear maps ϵ and

$n \mapsto n\big(\sum_{g\in\Pi} g\big)$. The resulting complex $\boldsymbol{T}$ of finite free R-modules is contractible over $\mathbb{Z}$ and hence so is the complex $\operatorname{Hom}_{\mathbb{Z}}(\boldsymbol{T},\mathbb{Z})$. The isomorphisms $\operatorname{Hom}_{\mathbb{Z}}(\boldsymbol{T},\mathbb{Z}) \cong \operatorname{Hom}_R(\boldsymbol{T},\operatorname{Hom}_{\mathbb{Z}}(R,\mathbb{Z})) \cong \operatorname{Hom}_R(\boldsymbol{T},R)$ show that $\boldsymbol{T}$ is totally acyclic. Thus, $\boldsymbol{T}$ is a Tate resolution of $\mathbb{Z}$ over R and $\widehat{\operatorname{Ext}}^n_R(\mathbb{Z},N) = \operatorname{H}^n \operatorname{Hom}_R(\boldsymbol{T},N)$ is the classical Tate cohomology group $\widehat{\operatorname{H}}^n(\Pi,N)$ of the group Π, cf. [29].

By Theorems (3.2.1) and (3.3.1), both relative cohomology and Tate cohomology are defined on the category $\widetilde{\mathcal{G}}$ of modules of finite G-dimension. The relations between these theories and absolute cohomology, provided by the morphisms $\varepsilon^n_{\mathcal{G}}$ of (2.1.2) and ε^n_R of (3.3), turn out to be unexpectedly tight.

3.3.4. Theorem (Avramov-Martsinkovsky [19]). *If* $\operatorname{G-dim}_R M = g < \infty$, *then for each R-module N there exist natural homomorphisms* $\delta^n_R(M,N)$, *such that the following sequence is exact:*

$$\begin{array}{l}
0 \longrightarrow \operatorname{Ext}^1_{\mathcal{G}}(M,N) \xrightarrow{\varepsilon^1_{\mathcal{G}}(M,N)} \operatorname{Ext}^1_R(M,N) \longrightarrow \quad \cdots \\
\longrightarrow \operatorname{Ext}^n_{\mathcal{G}}(M,N) \xrightarrow{\varepsilon^n_{\mathcal{G}}(M,N)} \operatorname{Ext}^n_R(M,N) \xrightarrow{\varepsilon^n_R(M,N)} \widehat{\operatorname{Ext}}^n_R(M,N) \\
\xrightarrow{\delta^n_R(M,N)} \operatorname{Ext}^{n+1}_{\mathcal{G}}(M,N) \longrightarrow \quad \cdots \quad \longrightarrow \widehat{\operatorname{Ext}}^g_R(M,N) \longrightarrow 0
\end{array}$$

The proof is based on a careful analysis of the homology exact sequence induced by the surjective morphism of complexes $\vartheta\colon \boldsymbol{T} \to \boldsymbol{P}$ from (3.3.1.iii).

3.4. Approximations. A *$\mathcal{G}$-approximation* of M is an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{G}$ and $\operatorname{proj\,dim}_R Y < \infty$. For existence of approximations we have:

3.4.1. Proposition. *M has a $\mathcal{G}$-approximation if and only if* $\operatorname{G-dim}_R M < \infty$.

Proof. If $\operatorname{G-dim}_R M < \infty$, then the strict $\mathcal{G}$-resolution from condition (3.2.1.ii) provides an exact sequence $0 \to \partial_1(G_1) \to G_0 \to M \to 0$ with $G_0 \in \mathcal{G}$ and $\operatorname{proj\,dim}_R(\partial_1(G_1)) < \infty$. Conversely, if $0 \to Y \to X \to M \to 0$ is a $\mathcal{G}$-approximation, then splice this exact sequence with a $\mathcal{P}$-resolution of Y of finite length to obtain a $\mathcal{G}$-resolution of finite length. □

Over local rings, G-dimension has additional properties.

3.5. Local rings. Let $(R,\mathfrak{m},k)$ be a local ring.

The next result gives the reason for the name *Gorenstein* dimension. The equivalence of (i) and (ii) follows from Theorems (1.4.1), (3.1.3), and (3.5.2).

3.5.1. Theorem (Auslander-Bridger [3]). *The following are equivalent.*

(i) *R is Gorenstein.*
(ii) $\operatorname{G-dim}_R N \le \operatorname{depth} R$ *for every finite R-module N.*
(iii) $\operatorname{G-dim}_R k < \infty$.

Finite projective dimension implies finite G-dimension by Theorem (3.1.2). It follows from the preceding theorem that the converse fails, so the next result properly contains Theorem (1.1.1).

3.5.2. Theorem (Auslander-Bridger [3]). *If* $\operatorname{G-dim}_R M < \infty$, *then*

$$\operatorname{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$$

For every finite R-module N, Corollary (3.1.2) and Theorem (B.1.1.2) yield

$$\operatorname{grade}_R(\operatorname{ann}_R(N), R) \leq \text{G-dim}_R N$$

If equality holds, then N is said to be *G-perfect*, by analogy with the notion described in (B.1.3); below we use the notation $N_R^\dagger$ also introduced there.

3.5.3. Theorem (Golod [44]). *If $R \to S$ is a surjective homomorphism of rings, and the R-module S is G-perfect with $S_R^\dagger \cong S$, then every finite S-module N satisfies*

$$\text{G-dim}_R N = \text{G-dim}_R S + \text{G-dim}_S N$$

This remarkable property of G-dimension is not shared by projective dimension: if $R = k[[t]]$, $S = R/(t^2)$, and $N = k$, then the complexes

$$0 \longrightarrow R \xrightarrow{\ t\ } R \longrightarrow 0$$

$$\cdots \longrightarrow S \xrightarrow{\ t\ } S \xrightarrow{\ t\ } S \xrightarrow{\ t\ } S \longrightarrow 0$$

provide minimal free resolutions of $N = k$ over R and over S, respectively.

Expositions. Chapters III and IV of the memoir of Auslander and Bridger [3] constitute the canonical text on Gorenstein dimension. There exist several accounts of that material within the framework of modules over local rings: in the early lectures of Auslander [2], the recent survey of Maşek [51], the monographs of Enochs and Jenda [36] and of Christensen [31]. The paper [19] provides a self-contained treatment of the classical results over arbitrary noetherian rings, using explicit constructions of complexes.

4. Minimal (co)resolutions

In this section R is an associative ring and M is an R-module.

We describe a concept of minimality from [19], which is applicable to every complex of R-modules. It explains the similarity of properties of minimal injective coresolutions and minimal projective resolutions, whenever the latter exist. It is also useful in situations where a notion of 'minimality' had been defined *ad hoc* (like maximal Cohen-Macaulay approximations), or has been missing (like proper resolutions). Omitted proofs can be found in [19].

4.1. Minimality. Let $\boldsymbol{B}$ be a complex of R-modules.

4.1.1. Proposition-Definition. *The following conditions are equivalent.*

(i) *Every homotopy equivalence $\beta\colon \boldsymbol{B} \to \boldsymbol{B}$ is an isomorphism.*
(ii) *Each morphism $\beta\colon \boldsymbol{B} \to \boldsymbol{B}$ homotopic to* $\operatorname{id}_{\boldsymbol{B}}$ *is an isomorphism.*
(iii) *If $\beta\colon \boldsymbol{B} \to \boldsymbol{C}$ and $\gamma\colon \boldsymbol{C} \to \boldsymbol{B}$ are homotopy inverses, then β is injective, γ is surjective,* $\operatorname{Ker}\gamma$ *is contractible, and* $\boldsymbol{C} = \operatorname{Im}\beta \oplus \operatorname{Ker}\gamma$.

When they hold, the complex $\boldsymbol{B}$ is said to ne minimal.

The proofs of the proposition and of its corollary use only basic homological algebra—the interpretation of homotopy classes of morphisms as homology classes in complexes of homomorphisms.

4.1.2. Corollary. *Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be minimal complexes of R-modules.*

(1) *Every homotopy equivalence $\beta\colon \boldsymbol{B} \to \boldsymbol{C}$ is an isomorphism.*
(2) *If $\boldsymbol{B}$ is minimal, and $\boldsymbol{A}$ is a contractible subcomplex of $\boldsymbol{B}$ such that for every $n \in \mathbb{Z}$ the R-module A_n is a direct summand of B_n, then $\boldsymbol{A} = 0$.*

A first example is psychologically satisfactory, but carries little substance.

4.1.3. Remark. Every complex concentrated in a single degree—in particular, every R-module—is a minimal complex.

Fortunately, less trivial examples of minimal complexes are at hand.

4.2. Minimal projective resolutions. We say that a projective resolution $\boldsymbol{P} \to M$ is *minimal* if the complex $\boldsymbol{P}$ is minimal. The existence of such resolutions depends on the availability of some special homomorphisms.

4.2.1. Projective covers. A *projective cover* of an R-module M is an epimorphism $\epsilon\colon P \to M$ from a projective R-module P, such that every homomorphism $\phi\colon P \to P$ with $\epsilon\phi = \epsilon$ is bijective. Over a general ring R, not every module has a projective cover.

4.2.2. Proposition. *Let $\boldsymbol{P}$ be a complex of projective R-modules.*

If $P_n \to \operatorname{Coker}(\partial_{n+1})$ is a projective cover for each n, then $\boldsymbol{P}$ is minimal; the converse holds when $\operatorname{Coker}(\partial_{n+1})$ has a projective cover for each n.

4.2.3. Corollary. *If $\boldsymbol{P} \to M$ is a minimal projective resolution, then*

$$\operatorname{proj\,dim}_R M = \sup\{n \in \mathbb{Z} \mid P_n \neq 0\}$$

If $\boldsymbol{P}' \to M$ is a projective resolution, then each comparison morphism $\boldsymbol{P}' \to \boldsymbol{P}$ is a split epimorphism with contractible kernel; it is bijective if $\boldsymbol{P}'$ is minimal.

Proofs may be obtained by dualizing those below. For finite modules over local rings, the general notion of minimal free resolution specializes to the standard concept introduced by Eilenberg [32]:

4.2.4. Example. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finite R-module.

By Nakayama's Lemma every finite projective R-module is free, an epimorphism $\epsilon\colon F \to M$ is a projective cover if and only if F is finite free and $\operatorname{Ker}(\epsilon) \subseteq \mathfrak{m}F$, and M has a projective cover. In view of these facts and the results above, a projective resolution $\boldsymbol{F} \to M$ is minimal if and only if each F_n is finite free and $\partial(\boldsymbol{F}) \subseteq \mathfrak{m}\boldsymbol{F}$, and every finite R-module has a unique up to isomorphism minimal free resolution.

4.3. Minimal injective coresolutions. We say that an injective coresolution $M \to \boldsymbol{I}$ is *minimal* if the complex $\boldsymbol{I}$ is minimal in the sense of (4.1).

4.3.1. Injective envelopes. An *injective envelope* of M is a monomorphism $\eta\colon M \to I$ to an injective module, such that each homomorphism $\iota\colon I \to I$ with $\iota\eta = \eta$ is bijective. Every R-module M has an injective envelope.

For injective coresolutions minimality coincides with the classical concept.

4.3.2. Proposition. *A complex $\boldsymbol{I}$ of injective modules is minimal if and only if $\operatorname{Ker}(\partial_n) \to I_n$ is an injective envelope for each $n \in \mathbb{Z}$.*

Proof. Assume first each I_n is an injective envelope of $\operatorname{Ker}(\partial_n)$, and let $\beta\colon \boldsymbol{I} \to \boldsymbol{I}$ be a morphism homotopic to $\operatorname{id}_{\boldsymbol{I}}$. We want to prove β_n is bijective. Since I_n is an injective envelope of $\operatorname{Ker}(\partial_n)$, it suffices to show that the restriction of β_n to $\operatorname{Ker}(\partial_n)$ is injective, that is, the module $K_n = \operatorname{Ker}(\partial_n) \cap \operatorname{Ker}(\beta_n)$ is trivial. Choose a homotopy θ from β to $\operatorname{id}_{\boldsymbol{I}}$. Every $x \in K_n$ satisfies $x = \partial_{n+1}\theta_n(x)$, hence $\theta_n(K_n) \cong K_n$ and $\theta_n(K_n) \cap \operatorname{Ker}(\partial_{n+1}) = 0$. Since I_{n+1} is an injective envelope of $\operatorname{Ker}(\partial_{n+1})$, we conclude $\theta_n(K_n) = 0$, so $K_n = 0$, as desired.

Assume next some I_n is not an injective envelope of $\operatorname{Ker}(\partial_n)$, so $I_n = I'_n \oplus J_n$ with I'_n an injective envelope of $\operatorname{Ker}(\partial_n)$ and $J_n \neq 0$. Thus, ∂_n maps J_n bijectively to a submodule J_{n-1} of I_{n-1}. Setting $J_i = 0$ for $i \neq n, n-1$ we get a contractible subcomplex $\boldsymbol{J} \subseteq \boldsymbol{I}$. As each J_i is injective, it splits off I_i, so $\boldsymbol{J}$ is irrelevant, hence $\boldsymbol{I}$ is not minimal by Proposition (4.1.3). □

4.3.3. Corollary. *If $M \to \boldsymbol{I}$ is a minimal injective coresolution, then*

$$\operatorname{inj\,dim}_R M = -\inf\{n \in \mathbb{Z} \mid I_n \neq 0\}$$

If $M \to \boldsymbol{I}'$ is an injective coresolution, then each comparison morphism $\boldsymbol{I}' \to \boldsymbol{I}$ is a split epimorphism with contractible kernel; it is bijective if $\boldsymbol{I}'$ is minimal.

Every R-module has a minimal injective coresolution.

Proof. Injective coresolutions of M are homotopy equivalent by the corresponding Comparison Lemma, so the first part follows from Corollary (4.1.2). For the second part, form an injective coresolution $M \to \boldsymbol{I}$ by choosing the maps $\eta\colon M \to I_0$, $\operatorname{Coker}\eta \to I_{-1}$, and $\operatorname{Coker}(\partial_{n+2}) \to I_n$ for all $n \leq -2$ to be injective envelopes. Proposition (4.3.2) shows that $\boldsymbol{I}$ is minimal. □

In the applications below the descriptions of minimal resolutions and the proofs of their existence are more delicate than in the classical cases. However, uniqueness properties are again obtained by specializing Corollary (4.1.2).

4.4. Minimal proper resolutions. We say that a $\mathcal{G}$-resolution $\gamma\colon \boldsymbol{G} \to M$ is *minimal* if $\boldsymbol{G}$ is a minimal complex.

4.4.1. Theorem. *Assume $(R, \mathfrak{m}, k)$ is a local ring and* $\operatorname{G-dim}_R M$ *is finite.*

The module M has a minimal $\mathcal{G}$-proper resolution. More precisely, a quasiisomorphism $\gamma\colon \boldsymbol{G} \to M$ is a minimal proper $\mathcal{G}$-resolution if and only if

(a) $G_n = 0$ *for all* $n < 0$.
(b) G_0 *is totally reflexive.*
(c) G_n *is free of finite rank for all* $n \geq 1$.
(d) $\partial_n(G_n)$ *contains no non-zero free direct summand of* G_{n-1} *for all* $n \geq 1$.

If $\boldsymbol{G} \to M$ is a minimal $\mathcal{G}$-proper resolution, then

$$\operatorname{G-dim}_R M = \sup\{n \in \mathbb{Z} \mid G_n \neq 0\}$$

If $\boldsymbol{G}' \to M$ is a $\mathcal{G}$-proper resolution, then each comparison morphism $\boldsymbol{G}' \to \boldsymbol{G}$ is a split epimorphism with contractible kernel; it is bijective if $\boldsymbol{G}'$ is minimal.

4.5. Minimal approximations. We say that a $\mathcal{G}$-approximation

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

is *minimal* if the complex $\cdots \to 0 \to Y \to X \to 0 \to \cdots$ is minimal.

4.5.1. Theorem. *Assume $(R, \mathfrak{m}, k)$ is a local ring and* $\operatorname{G-dim}_R M$ *is finite.*

The module M has a minimal $\mathcal{G}$-approximation. More precisely, an exact sequence $0 \to Y \xrightarrow{v} X \to M \to 0$ is a minimal $\mathcal{G}$-approximation if and only if X is totally reflexive, Y has finite projective dimension, and $v(Y)$ contains no non-zero direct summand of X.

If $0 \to Y' \to X' \to M \to 0$ is a $\mathcal{G}$-approximation, then it is isomorphic to

$$0 \longrightarrow Y \oplus F \xrightarrow{v \oplus \mathrm{id}_F} X \oplus F \longrightarrow M \longrightarrow 0$$

where F is a free R-module; it is minimal if and only if $F = 0$.

4.5.2. Remark. Let R be a Gorenstein local ring. A finite R-module M is totally reflexive if and only if it is *maximal Cohen-Macaulay*, that is, $\operatorname{depth}_R M = \dim R$. In particular, the $\mathcal{G}$-approximations of M are precisely the *maximal Cohen-Macaulay approximations* of Auslander and Buchweitz [7]. The theorem shows, in particular, that our concept of minimality for $\mathcal{G}$-approximations agrees with that of Auslander [8].

5. Betti numbers and Bass numbers

In this section $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module.

First we recall the definition of a sequences of homological invariants used to measure the size of a minimal projective resolution of M.

5.1. Betti numbers. By Example (4.2.4), every finite R-module M has a unique up to a isomorphism minimal resolution $\boldsymbol{F} \to M$. Thus,

$$\beta_n^R(M) = \operatorname{rank}_R(F_n)$$

is an invariant of M, called the nth *Betti number* of M over R.

The minimality of $\boldsymbol{F}$ is characterized by the condition $\partial(\boldsymbol{F}) \subseteq \mathfrak{m}\boldsymbol{F}$, so the complexes $\boldsymbol{F} \otimes_R k$ and $\operatorname{Hom}_R(\boldsymbol{F}, k)$ have trivial differentials. This leads to:

5.1.1. Remark. $\beta_n^R(M) = \operatorname{rank}_k \operatorname{Tor}_n^R(M, k) = \operatorname{rank}_k \operatorname{Ext}_R^n(M, k)$.

Corollary (4.2.4) also determines the non-vanishing Betti numbers:

5.1.2. Remark. $\beta_n^R(M) \neq 0$ if and only if $0 \leq n \leq \operatorname{proj\,dim}_R M$.

The equicharacteristic hypothesis appears in the next result because all known proofs use Hochster's 'big Cohen-Macaulay modules' [47]: they are known to exist only if R is an algebra over a field, or $\dim R \leq 2$.

5.1.3. Theorem (Evans-Griffith [37])**.** *If R is equicharacteristic and M has finite projective dimension, then there are inequalities*

$$\beta_n^R(M) \geq \begin{cases} 2n+1 & \text{for} \quad n = 0, \dots, \operatorname{proj\,dim}_R M - 2 \\ \operatorname{depth} R - \operatorname{depth}_R M & \text{for} \quad n = \operatorname{proj\,dim}_R M - 1 \\ 1 & \text{for} \quad n = \operatorname{proj\,dim}_R M \end{cases}$$

Examples of Bruns [24] show that the inequalities given by the syzygy theorem cannot be improved in general. On the other hand, much stronger bounds were conjectured in [27] for modules of finite length.

5.1.4. Conjecture (Buchsbaum-Eisenbud)**.** If $\operatorname{proj\,dim}_R M$ and $\operatorname{length}_R M$ are both finite, then the following inequalities hold

$$\beta_n^R(M) \geq \binom{\dim R}{n} \quad \text{for all} \quad n$$

Here are some comments on the status of this long open problem.

5.1.5. Remark. The conjectured bounds are best possible in general. Indeed, by the argument in Example (1.3.2) the existence of a module of finite length and finite projective dimension implies the existence of a module of finite length and finite injective dimension. Theorem (1.3.3) then shows that R is Cohen-Macaulay, so $\dim R = \operatorname{depth} R$ by Remark (B.3.3). For $d = \operatorname{depth} R$ the R-module N_d of Example (1.1.2) then has $\beta_R(N_d) = \binom{d}{n}$ for all n.

The conjecture holds trivially for $n = 0$. It holds for $n = 1$ by Krull's Theorem (B.3.1). As R is Cohen-Macaulay, the module M is perfect of finite length, hence so is the module $M^{\dagger}_{R}$. Thus, the conjecture holds for $M^{\dagger}_{R}$ and $n = 0, 1$. By Remark (5.3.3) below, it then holds for M and $n = d-1, d$.

The known cases imply the conjecture holds whenever $\dim R \leq 4$.

Next we describe sequences of invariants that measure the size of parts of a minimal injective resolution of M. There is one such sequence for each $\mathfrak{p} \in \operatorname{Spec} R$, where $\operatorname{Spec} R$ denotes the set of prime ideals of R. Invariants derived from injective resolutions are often more difficult to handle, due to the fact that injective modules are finite (if and) only if the ring R is artinian.

5.2. Bass numbers. Let $M \to \boldsymbol{I}$ be a minimal injective coresolution, cf. Corollary (4.3.3). By a well known theorem of Matlis, every injective module I_n is a direct sum of indecomposables, every indecomposable is isomorphic to the injective envelope $E_R(R/\mathfrak{p})$ for some $\mathfrak{p}$ in $\operatorname{Spec} R$, and the isomorphism

$$I_n \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu^n_R(\mathfrak{p}, M)}$$

uniquely defines $\mu^n_R(\mathfrak{p}, M)$, the nth *Bass number* of M over R at the prime $\mathfrak{p}$. When $\mathfrak{p} = \mathfrak{m}$, reference to the prime ideal is suppressed from terminology and notation.

The results on Betti numbers discussed above have analogs for Bass numbers. Proofs use the following expression, which is not hard to establish.

5.2.1. Lemma. $\mu^n_R(M) = \operatorname{rank}_k \operatorname{Ext}^n_R(k, M)$ *for all* $n \in \mathbb{Z}$.

Vanishing patterns for Bass numbers $\mu^n_R(M)$ are well understood. By Rees' Theorem (B.1.1.2), the first non-vanishing one appears for $n = \operatorname{depth}_R M$. Bass [22] proved the last non-vanishing number occurs for $n = \operatorname{inj\,dim}_R M$. The positivity of all intermediate $\mu^n_R(M)$, although formally similar to the statement for Betti numbers in Remark (5.1.2), is a more delicate fact.

5.2.2. Theorem (Fossum-Foxby-Griffith-Reiten [39]).

$$\mu^n_R(M) \neq 0 \quad \textit{if and only if} \quad \operatorname{depth}_R M \leq n \leq \operatorname{inj\,dim}_R M$$

In the next result, an equicharacteristic hypothesis on R appears for the same reason as in Theorem (5.1.3).

5.2.3. Theorem (Bruns [25]). *If R is equicharacteristic, then*

$$\mu^n_R(M) \geq \begin{cases} 1 & \textit{for} \quad n = \operatorname{depth}_R M \\ \operatorname{depth} R - \operatorname{depth}_R M & \textit{for} \quad n = \operatorname{depth}_R M + 1 \\ 2(\operatorname{depth} R - n) + 1 & \textit{for} \quad n = \operatorname{depth}_R M + 2, \dots, \operatorname{depth} R \end{cases}$$

5.3. Reductions. Certain adjustments can be made to facilitate computation of Betti numbers or Bass numbers. We provide some samples.

5.3.1. Remark. If $\varphi\colon R \to R'$ is a faithfully flat homomorphism of local rings, then $\beta^R_n(M) = \beta^{R'}_n(M \otimes_R R')$ for all $n \in \mathbb{Z}$.

Indeed, faithful flatness implies $\varphi(\mathfrak{m})$ is contained in the maximal ideal of R', so if $\boldsymbol{F} \to M$ is a minimal free resolution of M over R, then $\boldsymbol{F} \otimes_R R' \to M \otimes_R R'$ is a minimal free resolution of $M \otimes_R R'$ over R'.

5.3.2. Remark. $\beta_n^R(M) = \beta_n^{\widehat{R}}(\widehat{M})$ and $\mu_R^n(M) = \mu_{\widehat{R}}^n(\widehat{M})$ for all $n \in \mathbb{Z}$.

Indeed, $R \to \widehat{R}$ is faithfully flat by Theorem (B.4.1), so Remark (5.3.1) yields the equalities for Betti numbers. As $\mathfrak{m}\widehat{R}$ is the maximal ideal of $\widehat{R}$, we have $\operatorname{Ext}_R^n(k, M) \cong \operatorname{Ext}_{\widehat{R}}^n(k, M \otimes_R \widehat{R})$, whence the equalities for Bass numbers.

5.3.3. Remark. If M is perfect with $\operatorname{proj\,dim}_R M = g$, then

$$\beta_n^R(M) = \beta_{g-n}^R(M_R^\dagger) \quad \text{for all} \quad n \in \mathbb{Z}$$

Indeed, if $\boldsymbol{F} \to M$ is a minimal free resolution, then $\boldsymbol{F}^* \to M_R^\dagger$ is a free resolution by the definition of perfection (B.1.3); it is clearly minimal.

If a submodules is 'small' in the $\mathfrak{m}$-adic topology, then its Betti are tightly linked to those of the module and of the residue module.

5.3.4. Proposition (Avramov [9]). *There exists an integer s such that for each R-submodule L contained in $\mathfrak{m}^s M$ there is an equality*

$$\beta_n^R(M/L) = \beta_n^R(M) + \beta_{n-1}^R(L) \quad \textit{for all} \quad n \geq 0$$

Finally, we note that, most of the time, one can stay with Betti numbers.

5.3.5. Proposition (Foxby [41]). *For every finite R-module M there exists an $\widehat{R}$-module M' such that $\mu_n^R(M) = \beta_{n-d}^{\widehat{R}}(M')$ for all $n > d = \dim M$.*

Expositions. Proofs of the results in (5.1) and (5.2) are given in [26].

6. Complexity

In this section $(R, \mathfrak{m}, k)$ is a local ring and M a finite R-module.

We start a discussion of *asymptotic* properties of the sequences of Betti numbers and of Bass numbers for R-modules of infinite projective or injective dimension, to be continued in Sections 7 and 9. In view of Theorem (1.2.1), the only simple R-module, its residue field k, has has infinite classical dimensions whenever the ring R is *not* regular.

We describe a measure of the growth of Betti numbers, on the polynomial scale, introduced by Alperin and Evens [1] in a slightly different manner for finite dimensional representations of finite groups. The definition below is from [11]. For variants applying to general rings see the appendix of [12].

6.1. Complexity. The module M has *complexity* c, denoted $\operatorname{cx}_R M = c$, if c is the least non-negative integer d such that

$$\beta_n^R(M) \leq \beta n^{d-1} \quad \text{for a real number} \quad \beta \quad \text{and all} \quad n \gg 0$$

In particular, $\operatorname{cx}_R M = 0$ if and only if $\operatorname{proj\,dim}_R M < \infty$, so complexity offers a refinement of the dichotomy between finite and infinite projective dimension.

The first non-trivial results on projective dimension many a student sees concern the effect of factoring out a regular element. They have counterparts for complexity, but the next property of complexity has no antecedent.

6.1.1. Lemma. *If $R = Q/(x)$ where Q is a local ring and x is a Q-regular element, then*

$$\operatorname{cx}_Q M \leq \operatorname{cx}_R M \leq \operatorname{cx}_Q M + 1$$

Proof. Consider the classical change-of-rings spectral sequence

$$^2\mathrm{E}_{p,q} = \mathrm{Tor}^R_p\left(\mathrm{Tor}^Q_q(M,R),k\right) \implies \mathrm{Tor}^Q_{p+q}(M,k)$$

of Cartan-Eilenberg [29]. Isomorphisms

$$\mathrm{Tor}^Q_q(M,R) \cong \begin{cases} M & \text{if} \quad q=0,1 \\ 0 & \text{otherwise} \end{cases}$$

are easily obtained using the resolution $0 \to Q \xrightarrow{x} Q \to 0$ of R over Q. As $^2\mathrm{E}_{p,q} = 0$ for $q \neq 0,1$, the spectral sequence collapses to a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}^R_{n-1}(M,k) \longrightarrow \mathrm{Tor}^Q_n(M,k) \longrightarrow \mathrm{Tor}^R_n(M,k)$$
$$\xrightarrow{\chi_n} \mathrm{Tor}^R_{n-2}(M,k) \longrightarrow \mathrm{Tor}^Q_{n-1}(M,k) \longrightarrow \cdots$$

Estimates of vector space dimensions yield the desired inequalities. □

Clearly, the first (respectively, second) inequality in the lemma becomes an equality if the map χ_n in the exact sequence above is trivial for all n (respectively, surjective for all n). Sufficient conditions are given in the next result; the original proofs proceed by explicit constructions of a minimal resolution of M over R, starting from a minimal resolution of M over Q.

6.1.2. Theorem. *Assume $R = Q/(x)$, where $(Q,\mathfrak{n},k)$ is a local ring and x is a Q-regular element.*

(1) (Nagata [53]) *If $x \notin \mathfrak{n}^2$, then $\mathrm{cx}_R M = \mathrm{cx}_Q M$.*
(2) (Shamash [65]) *If $x \in \mathfrak{n}\,\mathrm{ann}_Q M$, then $\mathrm{cx}_R M = \mathrm{cx}_Q M + 1$.*

Complete intersections take their turn for a homological characterization.

6.2. Complete intersection local rings. The next characterization of the complete intersection property parallels that of regularity in Theorem (1.2.1).

6.2.1. Theorem (Gulliksen [46]). *The following are equivalent.*

(i) *R is complete intersection.*
(ii) $\mathrm{cx}_R N \le \mathrm{edim}\, R - \mathrm{depth}\, R$ *for every finite R-module N.*
(iii) $\mathrm{cx}_R k < \infty$.

Proof. (i) $\implies$ (ii). Remark (5.3.2) yields $\mathrm{cx}_R N = \mathrm{cx}_{\widehat{R}}(N \otimes_R \widehat{R})$, while for trivial reasons we have $\mathrm{edim}\,\widehat{R} = \mathrm{edim}\, R$ and $\mathrm{depth}\,\widehat{R} = \mathrm{depth}\, R$. Thus, we may assume R is complete, and hence $R \cong Q/(\boldsymbol{x})$ for a regular local ring with $\mathrm{edim}\, Q = \mathrm{edim}\, R$ and a Q-regular sequence $\boldsymbol{x}$, cf. Remark (B.4.4). Theorem (1.2.1) yields $\mathrm{cx}_Q N = 0$, so Lemma (6.1.1) gives the inequality below:

$$\begin{aligned} \mathrm{cx}_R N \le \mathrm{card}(\boldsymbol{x}) &= \mathrm{depth}\, Q - \mathrm{depth}\, R \\ &= \mathrm{edim}\, Q - \mathrm{depth}\, R = \mathrm{edim}\, R - \mathrm{depth}\, R \end{aligned}$$

The equalities come from the definitions of depth and of regularity.

(iii) $\implies$ (i) is proved by very different arguments, cf. Section 7. □

As already noted, a module has complexity 0 if and only if it has finite projective dimension, that is, its Betti numbers eventually vanish. The next simplest class, that of complexity at most 1, consists of those modules whose Betti numbers are bounded. Over complete intersections the minimal resolution of such a module is described by

6.2.2. Theorem (Eisenbud [34]). *If R is complete intersection, $\operatorname{cx}_R M \leq 1$ and $\boldsymbol{F} \to M$ is a minimal free resolution, then $\boldsymbol{F}$ has the form*

$$\cdots \xrightarrow{\beta} G \xrightarrow{\alpha} F \xrightarrow{\beta} G \xrightarrow{\alpha} F \xrightarrow{\partial_{g+1}} F_g \xrightarrow{\partial_g} F_{g-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

where $g = \operatorname{depth} R - \operatorname{depth} M$ and F, G are free R-modules of the same rank.

Next we describe complete intersections in the spirit of Theorem (1.1.1).

6.3. Lower complete intersection dimension. Let $(R, \mathfrak{m}, k)$ be a local ring. Recall from Section 3 that $\mathcal{G}$ denotes the category of totally reflexive R-modules, and let $\mathcal{B}$ denote its subcategory consisting of those modules that have finite complexity. Clearly, $\mathcal{B}$ contains all finite free R-modules. Using free resolutions constructed via mapping cones and the Horseshoe Lemma, and the fact that $\mathcal{G}$ is a resolving subcategory for the category $\mathcal{F}$ of finite R-modules, cf. Lemma (3.1.1), one obtains:

6.3.1. Lemma. *The subcategory $\mathcal{B}$ is resolving for $\mathcal{F}$.*

Following [43] (but changing terminology and notation), we use (2.2.2) to define a *lower complete intersection dimension* of M over R by the formula

$$\text{CI}_*\text{-}\dim_R M = \mathcal{B}\text{-}\dim(M)$$

6.3.2. Theorem (Gerko [43]). *The following are equivalent.*

(i) *R is complete intersection.*
(ii) $\text{CI}_*\text{-}\dim_R N \leq \operatorname{depth} R$ *for every finite R-module N.*
(iii) $\text{CI}_*\text{-}\dim_R k < \infty$.

Proof. Assume first R is complete intersection. By Remark (B.2.1) it is Gorenstein, so for every finite R-module N Theorem (3.5.1) yields $\text{G-}\dim_R N \leq d$ where $d = \operatorname{depth} R$. If $\boldsymbol{F} \to N$ is a minimal free resolution, then the R-module $\Omega^d \boldsymbol{F}$ is totally reflexive by Theorem (2.2.2). On the other hand, $\operatorname{cx}_R(\Omega^d \boldsymbol{F}) = \operatorname{cx}_R N < \infty$, with the equality holding by definition, and the inequality by Theorem (6.2.1). Thus, we have shown $\Omega^d \boldsymbol{F}$ is in $\mathcal{B}$, hence $\text{CI}_*\text{-}\dim_R N \leq d$ by Proposition (2.2.2).

Assume next $\text{CI}_*\text{-}\dim_R k = g < \infty$. If $\boldsymbol{P} \to k$ is a free resolutions, then $\Omega^g \boldsymbol{P}$ is in $\mathcal{B}$ by Lemma (6.3.1) and Proposition (2.2.2), hence $\operatorname{cx}_R k = \operatorname{cx}_R(\Omega^g \boldsymbol{P}) < \infty$. From Theorem (6.2.1) we conclude that R is complete intersection. □

Modules of finite lower complete intersection dimension over general rings may lack some of the nice properties of modules over complete intersections. One such discrepancy can be seen by comparing Theorem (6.2.2) with the following counterexample to a conjecture of Eisenbud [34].

6.3.3. Example (Gasharov-Peeva [42]). Let k be a field, let $t_1, \ldots, t_4$ be indeterminates over k, fix $a \in k \smallsetminus \{0\}$, and set

$$R_{(a)} = \frac{k[t_1, t_2, t_3, t_4]}{(at_1t_3 + t_2t_3\,,\ t_1t_4 + t_2t_4\,,\ t_3t_4\,,\ t_1^2\,,\ t_2^2\,,\ t_3^2\,,\ t_4^2)}$$

It is easy to check that the minimal complex of $R_{(a)}$-modules

$$\boldsymbol{T}_{(a)} = \quad \cdots \longrightarrow R_{(a)}^2 \xrightarrow{\begin{pmatrix} t_1 & a^n t_3 + t_4 \\ 0 & t_2 \end{pmatrix}} R_{(a)}^2 \longrightarrow \cdots$$

is exact, and so is the complex $(\boldsymbol{T}_{(a)})^*$. Thus, the $R_{(a)}$-module

$$M_{(a)} = \operatorname{Coker}\begin{pmatrix} t_1 & t_3 + t_4 \\ 0 & t_2 \end{pmatrix}$$

has $\beta_n^{R_{(a)}}(M_{(a)}) = 2$ for all $n \geq 0$, and is in $\mathcal{G}$ by Theorem (3.3.1).

It is proved in [42] that $\boldsymbol{T}_{(a)}$ is periodic of period q if and only if q is the multiplicative order of a. Thus, k and a may be chosen so that the artinian (and hence Cohen-Macaulay) ring $R_{(a)}$ has a module $M_{(a)}$ with $\operatorname{CI}_*\text{-}\dim_{R_{(a)}} M_{(a)} = 0$ and $\operatorname{cx}_{R_{(a)}} M_{(a)} = 1$, whose minimal resolution is periodic of period q for any integer $q \geq 1$, or is not periodic.

Similar examples are constructed in [42] also over Gorenstein rings.

Expositions. The survey [14] contains proofs for the results in this section.

7. Complete intersection dimension

In this section $(R, \mathfrak{m}, k)$ is a local ring and M a finite R-module.

We describe a dimension whose finiteness for all modules characterizes complete intersection rings, and whose finiteness for an individual module imposes strict conditions on the structure of its minimal resolutions.

7.1. Complete intersection dimension. A diagram of homomorphisms $R \to R' \leftarrow Q$ of local rings is a *quasi-deformation* if $R \to R'$ is faithfully flat and $R' \leftarrow Q$ is surjective with kernel generated by a regular sequence. We set

$$\operatorname{CI-dim}_R M = \inf\left\{ \operatorname{proj\,dim}_Q (M \otimes_R R') - \operatorname{proj\,dim}_Q R' \;\middle|\; \begin{matrix} R \to R' \leftarrow Q \text{ is a} \\ \text{quasi-deformation} \end{matrix} \right\}$$

This invariant, introduced in [17], is called the *complete intersection dimension*, or *CI-dimension*, of M over R. Here is one of its basic properties.

7.1.1. Remark. If $\operatorname{CI-dim}_R M$ is finite, then so is $\operatorname{cx}_R M$ by Lemma (5.3.1) and Remark (6.1.1). In fact, there is a common upper bound on the complexities of *all* R-modules of finite CI-dimension, cf. Theorem (7.3.3).

The next easy consequence of Theorem (6.2.1) is noted in [11], [17].

7.1.2. Theorem. *The following are equivalent.*

(i) *R is complete intersection.*
(ii) $\operatorname{CI-dim}_R N \leq \operatorname{depth} R$ *for every finite R-module N.*
(iii) $\operatorname{CI-dim}_R k < \infty$.

Proof. (i) $\implies$ (ii) Remark (B.4.4) yields a quasi-deformation $R \to \widehat{R} \leftarrow Q$ with Q a regular local ring, so $\operatorname{proj\,dim}_Q(N \otimes_R \widehat{R}) < \infty$ by Theorem (1.2.1).

(iii) $\implies$ (i) Remark (7.1.1) yields $\operatorname{cx}_R k < \infty$, so R is complete intersection by Theorem (6.2.1). □

The study of modules of finite CI-dimension depends on the use of classical multiplicative structures carried by homology and cohomology over such rings.

7.2. Cohomology operators. Let Q be a commutative ring, let $\boldsymbol{x} = x_1, \dots, x_c$ be a Q-regular sequence, set $J = (\boldsymbol{x})$, and $R = Q/J$. For the canonical action of R, the module J/J^2 is free and the residue classes $x_1 + J^2, \dots, x_c + J^2$ form a basis.

Let $x_1^*, \dots, x_c^*$ denote the dual basis of the free R-module $\operatorname{Hom}_R(J/J^2, R)$. We identify the symmetric algebra of this module with the polynomial ring

$$S = R[x_1^*, \dots, x_c^*]$$

graded by assigning *cohomological degree* 2 to the indeterminates. We call $x_1^*, \dots, x_c^*$ the *cohomology operators* defined by the regular sequence $\boldsymbol{x}$.

The next result subsumes the work of several authors, cf. [21] for details, and (A.3) for the algebra and module structures in cohomology used in it.

7.2.1. Theorem (Avramov-Sun [21]). *For every R-module N there exist homomorphisms of graded R-algebras*

$$\operatorname{Ext}_R^{\bullet}(N,N) \xleftarrow{\zeta_N} R[x_1^*, \dots, x_c^*] \xrightarrow{\zeta_M} \operatorname{Ext}_R^{\bullet}(M,M)$$

with images in the centers of the respective Ext *algebras, and such that*

$$\zeta_N(x_i^*)\cdot\varepsilon = \varepsilon\cdot\zeta_M(x_i^*) \quad \text{for all} \quad \varepsilon\in\operatorname{Ext}_R^{\bullet}(M,N) \quad \text{and for} \quad i=1,\dots,c$$
$$\zeta_N(x_i^*)\cdot\tau = -\zeta_M(x_i^*)\cdot\tau \quad \text{for all} \quad \tau\in\operatorname{Tor}_{\bullet}^R(M,N) \quad \text{and for} \quad i=1,\dots,c$$

For applications, it is crucial to understand the non-triviality of the action of the polynomial ring, resulting from the preceding result.

7.2.2. Theorem (Gulliksen [45]; Avramov-Gasharov-Peeva [17]). *The $R[x_1^*, \dots, x_c^*]$-module $\operatorname{Ext}_R^{\bullet}(M,N)$ is finite if and only if $\operatorname{Ext}_Q^n(M,N) = 0$ for all $n \gg 0$.*

Combining the preceding results, one gets

7.2.3. Corollary. *If $\text{CI-dim}_R M < \infty$, then $\operatorname{Ext}_R^{\bullet}(M,k)$ is a finite graded module over a polynomial ring generated by central elements in $\operatorname{Ext}_R^2(k,k)$.*

One of the reasons for introducing CI-dimension was to find, over arbitrary rings, modules whose resolutions have patterns of growth similar to that of resolutions over complete intersection rings.

7.3. Betti numbers. Assume $\text{CI-dim}_R M$ is finite. The proofs of the next two theorems use the results in (7.2). The first one extends Theorem (6.2.2).

7.3.1. Theorem (Avramov [11]). *If $\operatorname{cx}_R M \le 1$, then each minimal free resolution $\boldsymbol{F} \to M$ is periodic of period 2 after $\operatorname{depth} R - \operatorname{depth} M$ steps and*

$$\beta_{n+1}^R(M) = \beta_n^R(M) \quad \text{for all} \quad n > \operatorname{depth} R - \operatorname{depth} M$$

7.3.2. Theorem (Avramov-Gasharov-Peeva [17]). *If $\operatorname{cx}_R M = d \ge 2$, then there exist integers $b > 0$ and $0 \le e \le d-1$ and polynomials*

$$b_{\pm}(t) = \frac{b}{2^e(d-1)!}t^{d-1} + \text{ lower order terms } \in \mathbb{Q}[t]$$

such that for all $n \gg 0$ there are equalities

$$\beta_n^R(M) = \begin{cases} b_+(n) & \text{when } n \text{ is even} \\ b_-(n) & \text{when } n \text{ is odd} \end{cases}$$

Furthermore, both difference polynomials $b_{\pm}(t+1) - b_{\mp}(t)$ have degree $d-2$ and positive (but possibly different) leading coefficient, hence there are inequalities

$$\beta_{n+1}^R(M) > \beta_n^R(M) \quad \text{for all} \quad n \gg 0$$

The next result extends the implication (i) $\implies$ (ii) of Theorem (6.2.1).

7.3.3. Theorem (Avramov-Gasharov-Peeva [17]). *There is an inequality*

$$\operatorname{cx}_R M \leq \operatorname{edim} R - \operatorname{depth} R$$

Over complete intersections the inequality above follows from the fairly elementary Lemma (6.1.1), but the proof of the general case relies on deep structural properties of the algebra $\operatorname{Ext}_R^{\bullet}(k,k)$, cf. Remark (9.4).

7.4. Support varieties. Let $(R, \mathfrak{m}, k)$ be a local complete intersection with

$$\operatorname{edim} R - \operatorname{depth} R = c$$

Let $\widetilde{k}$ denote an algebraic closure of k, and let M, N be finite R-modules.

By Theorem (7.2.1), $\operatorname{Ext}_R^{\bullet}(M,N) \otimes_R k$ is a finite graded module over the graded polynomial ring $k[x_1^*, \dots, x_c^*]$. The *support variety* of the pair (M,N) is the algebraic set $\operatorname{V}_R^*(M,N) \subseteq \widetilde{k}^c$ defined by the vanishing of all polynomials $f(x_1^*, \dots, x_c^*)$ that annihilate $\operatorname{Ext}_R^{\bullet}(M,N) \otimes_R k$. The annihilator of a graded module is a graded ideal, so $\operatorname{V}_R^*(M,N)$ is a cone with vertex at the origin. We set $\operatorname{V}_R^*(M) = \operatorname{V}_R^*(M,k)$. The construction in two variables was introduced in [15]. It extends a direct construction of $\operatorname{V}_R^*(M)$ in [11], itself partly inspired by support varieties associated to representations of finite groups; Benson's book [23] provides a systematic account of the use of varieties in representation theory. We list three (among many) properties of support varieties.

7.4.1. Theorem (Avramov-Buchweitz [15]). *Support varieties satisfy:*

(1) $\operatorname{V}_R^*(M,M) = \operatorname{V}_R^*(M) = \operatorname{V}_R^*(k,M)$.

(2) $\operatorname{V}_R^*(M,N) = \{0\}$ *if and only if* $\operatorname{Ext}_R^n(M,N) = 0$ *for* $n \gg 0$.

(3) $\operatorname{V}_R^*(M,N) = \operatorname{V}_R^*(M) \cap \operatorname{V}_R^*(N) = \operatorname{V}_R^*(N,M)$.

The dimension of the variety $\operatorname{V}_R^*(M,N)$ measures 'the size' of $\operatorname{Ext}_R^*(M,N)$. To make this precise, we define the *complexity* of the pair (M,N) by

$$\operatorname{cx}_R(M,N) = \inf \left\{ b \in \mathbb{N} \;\middle|\; \begin{array}{c} \nu_R\big(\operatorname{Ext}_R^n(M,N)\big) \leq a\,n^{b-1} \text{ for some} \\ \text{real number } a \text{ and for all } n \gg 0 \end{array} \right\}$$

where $\nu_R(E)$ denotes the minimal number of generators of an R-module E. Elementary facts from dimension theory of algebraic varieties give

$$\operatorname{cx}_R(M,N) = \dim \operatorname{V}_R^*(M,N)$$

On the other hand, $\operatorname{cx}_R(M,k) = \operatorname{cx}_R M$ holds by definition, so basic results on intersection of homogeneous varieties turn the structural statements of the preceding theorem into numerical data on the growth of resolutions.

7.4.2. Corollary. *The following (in)equalities hold:*

(1) $\operatorname{cx}_R(M,M) = \operatorname{cx}_R M$.

(2) $\operatorname{cx}_R(M,N) = 0$ *if and only if* $\operatorname{Ext}_R^n(M,N) = 0$ *for* $n \gg 0$.

(3) $\operatorname{cx}_R M + \operatorname{cx}_R N - \operatorname{codim} R \leq \operatorname{cx}_R(M,N)$
$= \operatorname{cx}_R(N,M) \leq \min\{\operatorname{cx}_R M, \operatorname{cx}_R N\}$.

The statements above are purely algebraic, but their proofs—and even the guess that they might hold—came from the geometric intuition provided by support varieties. The same remark applies to the following vanishing theorem for (co)homological functors over complete intersections.

7.4.3. Theorem (Avramov-Buchweitz [15]). *The following are equivalent.*

(i) $\mathrm{Ext}^n_R(M,N)=0$ *for* $h\le n\le h+c$ *and some* $h>\dim R$.

(ii) $\mathrm{Ext}^n_R(N,M)=0$ *for* $i\le n\le i+c$ *and some* $i>\dim R$.

(iii) $\mathrm{Tor}^R_n(M,N)=0$ *for* $j\le n\le j+c$ *and some* $j>0$.

(iv) $\mathrm{Ext}^n_R(M,N)=\mathrm{Ext}^n_R(N,M)=\mathrm{Tor}^R_n(M,N)=0$ *for all* $n>\dim R$.

7.4.4. Remark. The theorem above yields an unexpected and remarkable feature of complete intersection local rings:

(**ee**) Every pair (M,N) of finite R-modules has the following property: if $\mathrm{Ext}^n_R(M,N)=0$ for all $n\gg 0$ then $\mathrm{Ext}^n_R(N,M)=0$ for all $n\gg 0$.

On the other hand, if condition (**ee**) holds for a local ring $(R,\mathfrak{m},k)$, then applied to the pair (R,k) it yields $\mu^n_R(R)=0$ for all $n\gg 0$, so R is Gorenstein by Theorems (5.2.2) and (1.4.1). Thus, the class of rings satisfying condition (**ee**) lies between the complete intersection rings and the Gorenstein rings, cf. (B.2.1). Very recently, Huneke-Jorgensen [48] and Şega [63] have provided examples of non-complete-intersection Gorenstein rings that satisfy (**ee**). It is not known whether all Gorenstein rings have this property.

8. Dimensions from deformations

In this section $(R,\mathfrak{m},k)$ denotes a local ring and M a finite R-module.

We describe new homological dimensions whose definitions involve two choices—that of a classical homological dimension, and that of a class of ring homomorphisms allowing descent without increasing the classical dimension. The procedure extends the one used for CI-dimension in (7.1). It partly formalizes the existence of Cohen presentations from Theorem (B.4.3), and the characterizations of properties of residue rings of regular local rings in terms of the defining ideal from Theorem (B.2.2).

8.1. Resolving categories and homomorphisms. Let $\mathcal{C}(R)$ be a resolving subcategory of the category $\mathcal{F}(R)$ a finite R-modules, for which the following holds: Whenever $R\to R'$ is a faithfully flat homomorphism of local rings, M belongs to $\mathcal{C}(R)$ if and only if $M\otimes_R R'$ belongs to $\mathcal{C}(R')$. The categories $\mathcal{P}(R)$ of projective R-modules, $\mathcal{G}(R)$ of totally reflexive R-modules, and $\mathcal{B}(R)$ of totally reflexive R-modules of finite complexity all have this property.

Fixing a class h of surjective homomorphisms of local rings, we say that a diagram of local homomorphisms $R\to R'\leftarrow Q$ is a *h-quasi-deformation* if $R\to R'$ is flat extension and $R'\leftarrow Q$ belongs to h. We now set

$$h\|\mathcal{C}\text{-}\dim M=\inf\left\{\mathcal{C}(Q)\text{-}\dim\,(M\otimes_R R')-\mathcal{C}(Q)\text{-}\dim R'\;\middle|\;\begin{matrix}R\to R'\leftarrow Q\text{ is an}\\ h\text{-quasi-deformation}\end{matrix}\right\}$$

Note that if h consists of isomorphisms, then $h\|\mathcal{C}\text{-}\dim M=\mathcal{C}\text{-}\dim M$, so the new batch of dimensions comprises those defined solely in terms of resolutions.

The dimension described in (7.1) also fits this mold.

8.2. Complete intersection dimension. If c denotes the class of surjective homomorphisms with kernel generated by a regular sequence, then

$$\text{CI-}\dim_R M=c\|\mathcal{P}\text{-}\dim M$$

To introduce the next concept we use the notation of (B.1.3).

8.3. Upper Gorenstein dimension. Let g denote the class of surjective homomorphisms $Q \to R'$ such that R' is a perfect Q-module with ${R'}^{\dagger}_{Q} \cong R'$.

Veliche [68] introduces the *upper Gorenstein dimension* of M over R by

$$\mathrm{G}^{*}\text{-}\dim_R M = g\|\mathcal{P}\text{-}\dim M$$

Below we use the notion of G-perfect module from (3.5).

8.4. Cohen-Macaulay dimension. Let p denote the class of all surjective homomorphisms $Q \to R'$ making R' into a G-perfect Q-module.

Gerko [43] defines the *Cohen-Macaulay dimension* of M over R to be

$$\mathrm{CM}\text{-}\dim_R M = p\|\mathcal{G}\text{-}\dim M$$

The new homological dimensions can be computed from *any* h-quasi-deformation that establishes its finiteness: this follows from standard change-of-rings results for depth, projective dimension, and G-dimension:

8.5. Lemma. *If* $(h\|\mathcal{C})$ *equals* $(c\|\mathcal{P})$ *(respectively,* $(g\|\mathcal{P})$ *or* $(p\|\mathcal{G})$*) and there is a* h*-quasi-deformation* $R \to R' \leftarrow Q$ *with* $\mathcal{C}(Q)\text{-}\dim(M \otimes_R R') < \infty$*, then*

$$h\|\mathcal{C}\text{-}\dim M = \mathcal{C}(Q)\text{-}\dim(M \otimes_R R') - \mathcal{C}(Q)\text{-}\dim R'$$

We now have a *homological dimension* of M over R, denoted $\mathrm{H}\text{-}\dim_R M$, defined for $\mathrm{H} = \mathrm{CI}$ in (7.1), $\mathrm{H} = \mathrm{CI}_*$ in (6.3), $\mathrm{H} = \mathrm{G}$ in (3.1), $\mathrm{H} = \mathrm{G}^*$ in (8.3), and $\mathrm{H} = \mathrm{CM}$ (8.4). For uniformity of notation, in this section we set $\mathrm{Reg}\text{-}\dim M = \operatorname{proj\,dim}_R M$. We say that the ring R has property Reg (respectively, CI, CI_*, G, G^*, CM) if it is regular (respectively, complete intersection, complete intersection, Gorenstein, Gorenstein, Cohen-Macaulay).

A feature common to all these dimensions is recorded in the next theorem: cf. Theorems (1.2.1), (7.1.2), (6.3.2), and (3.5.1) for the first four values of H, [68] for $\mathrm{H} = \mathrm{G}^*$, and [43] for $\mathrm{H} = \mathrm{CM}$.

8.6. Theorem. *For* H *equal to* Reg *(respectively,* CI, CI_*, G, G^*, CM*) the following conditions are equivalent.*

(i) *R has property* H.
(ii) $\mathrm{H}\text{-}\dim_R N \le \operatorname{depth} R$ *for every finite R-module N.*
(iii) $\mathrm{H}\text{-}\dim_R k < \infty$.

In an omnibus statement, we collect properties common to most dimensions. They are textbook fare for H = Reg and come from [3] for H = G; once again, proofs for the remaining values of H are given in [17], [43], [68].

8.7. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

For H *equal to* Reg *(respectively,* CI, CI_*, G^*, G*, or* CM*) the homological dimension* $\mathrm{H}\text{-}\dim_R(M)$ *has the following properties.*

(0) $\mathrm{H}\text{-}\dim_R M = -\infty$ *if and only if* $M = 0$.
(1) $\mathrm{H}\text{-}\dim_R M = \operatorname{depth} R - \operatorname{depth}_R M$ *if* $\mathrm{H}\text{-}\dim_R M < \infty$.
(2) $\mathrm{H}\text{-}\dim_R M \le \mathrm{H}\text{-}\dim_R k$.

If $\boldsymbol{F} \to M$ is a free resolution, then

(3) $\mathrm{H}\text{-}\dim_R M = \max\{\mathrm{H}\text{-}\dim_R(\Omega^1 \boldsymbol{F}) - 1, 0\}$ *if* $M \ne 0$.
(4) $\mathrm{H}\text{-}\dim_R M \ge \mathrm{H}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p}$ *for every prime ideal* $\mathfrak{p}$ *of* R.
(5) $\mathrm{H}\text{-}\dim_R M = \mathrm{H}\text{-}\dim_{\widehat{R}}(M \otimes_R \widehat{R})$.

If a local ring S is a faithfully flat R-algebra, then $M' = M \otimes_R S$ satisfies

(6) $\text{H-dim}_R M \le \text{H-dim}_S M'$
with equality if $\text{H-dim}_S(M') < \infty$ *or* $\text{H} \in \{\text{Reg}, \text{CI}_*, \text{G}\}$.

For an element $y \in \mathfrak{m}$ the module $\overline{M} = M/yM$ and ring $\overline{R} = R/(y)$ satisfy

(7) $\text{H-dim}_R M = \text{H-dim}_R(\overline{M}) - 1$ *when y is M-regular.*

(8) $\text{H-dim}_R M \ge \text{H-dim}_{\overline{R}}(\overline{M})$ *when y is M-regular and R-regular*
with equality if $\text{H-dim}_R M < \infty$ *or* $\text{H} \in \{\text{Reg}, \text{CI}_*, \text{G}\}$.

(9) $\text{H-dim}_R M \ge \text{H-dim}_{\overline{R}} M + 1$ *when y is R-regular,* $yM = 0$, $\text{H} \ne \text{Reg}$
with equality if $\text{H-dim}_R M < \infty$ *or* $\text{H} \in \{\text{CI}_*, \text{G}\}$.

$\text{H-dim}_R M \le \text{H-dim}_{\overline{R}} M + 1$ *when y is R-regular,* $yM = 0$, $\text{H} = \text{Reg}$
with equality if $\text{H-dim}_{\overline{R}} M < \infty$.

The two parts of (9) underscore the difference between projective and Gorenstein dimensions, recorded in Theorem (3.5.3). It is desirable to know if equalities always hold in (6), (8), and (9) for H equal to CI, G^*, or CM, and whether these dimensions satisfy the inequality in Proposition (2.2.4). Next we bring the hierarchy of local rings (B.2.1) to the level of individual modules.

8.8. Theorem. *Homological dimensions satisfy the inequalities*

$$\text{CM-dim}_R M \le \text{G-dim}_R M \le \text{CI}_*\text{-dim}_R M \le \text{CI-dim}_R M \le \operatorname{proj\,dim}_R M$$
$$\text{G-dim}_R M \le \text{G}^*\text{-dim}_R M \le \text{CI-dim}_R M$$

If any one of these dimensions is finite, then it is equal to those to its left.

The inequalities follow from the definitions, except for the upper bound for $\text{CI}_*\text{-dim}_R M$ and the lower bound for $\text{G}^*\text{-dim}_R M$; they are proved in [17] and [68], respectively. The inequalities imply that if one of these dimensions is finite, then so are those to its left, which then are equal by Theorem (8.7.1). The next result illustrates the usefulness of various dimensions for classifying modules over general local rings (a different proof of (1) is given in [17]).

8.9. Theorem (Veliche [68]). *Let $(Q, \mathfrak{n}, k)$ be a local ring, which in part (n) below is assumed to satisfy* $\operatorname{depth} Q \ge n$. *There exists then a pair (J, N), where J is ideal contained in $\mathfrak{n}^2$ such that the Q-module $R = Q/J$ is perfect and N is a finite R-module, with the following property:*

(1) $\operatorname{grade}_Q R \ge 1$ *and* $0 = \text{CI-dim}_R N < \operatorname{proj\,dim}_R N = \infty$.
(2) $\operatorname{grade}_Q R = 2$ *and* $0 = \text{CM-dim}_R N < \text{G-dim}_R N = \infty$.
(3) $\operatorname{grade}_Q R = 3$, *and* $0 = \text{G}^*\text{-dim}_R N < \text{CI}_*\text{-dim}_R N = \infty$.

If Q is equicharacteristic, then furthermore

(4) $\operatorname{grade}_Q R = 4$ *and* $0 = \text{CI}_*\text{-dim}_R N < \text{CI-dim}_R N = \infty$.
(5) $\operatorname{grade}_Q R = 5$ *and* $0 = \text{G}^*\text{-dim}_R M = \text{CI}_*\text{-dim}_R N < \text{CI-dim}_R N = \infty$.

No example of module N with $\text{CI}_*\text{-dim}_R N < \text{G}^*\text{-dim}_R N$ or, more generally, with $\text{G-dim}_R N < \text{G}^*\text{-dim}_R N$ is known at present.

9. Curvature

In this section $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module.

We first note that no sequence of Betti numbers can grow faster than exponentially, then exhibit classes of modules whose Betti sequences admit also an exponential lower bound. Many results on exponential growth have counterparts in rational homotopy theory, cf. Remark (9.4).

9.1. Upper bounds. We start with an easy computation.

9.1.1. Example. If R is artinian and $\operatorname{length}_R R = l$, then

$$\beta_n^R(M) \leq \beta_0^R(M)(l-1)^n \quad \text{for each} \quad n \geq 1$$

Indeed, if $\boldsymbol{F} \to M$ is a minimal free resolution, then $F_n/\mathfrak{m}F_n \cong (\Omega^n \boldsymbol{F})/\mathfrak{m}(\Omega^n \boldsymbol{F})$ for $n \geq 0$ and $\Omega^n \boldsymbol{F} \subseteq \mathfrak{m}F_{n-1}$ for all $n \geq 1$, hence

$$\begin{aligned}\beta_n^R(M) = \operatorname{length}(F_n/\mathfrak{m}F_n) &\leq \operatorname{length}_R(\Omega^n \boldsymbol{F}) \\ &\leq \operatorname{length}(\mathfrak{m}F_{n-1}) = (l-1)\beta_{n-1}^R(M)\end{aligned}$$

Iterating the inequality above, we obtain the desired exponential bound.

For non-artinian rings, the first inequality in the next proposition is somewhat harder to establish. For the second inequality, note that Proposition (5.3.4) with $L = \mathfrak{m}^s M$ yields $\beta_n^R(M) \leq \beta_n^R(M/\mathfrak{m}^s M)$ for all n, so it suffices to prove the assertion for modules of finite length. In that case a simple induction on $\operatorname{length}_R M$ shows that the desired inequality holds with $\lambda = \operatorname{length}_R M$.

9.1.2. Proposition. *The Betti numbers of M are bounded above as follows.*

(1) $\beta_n^R(M) \leq \alpha^n$ *for some real number α and each $n \geq 1$.*
(2) $\beta_n^R(M) \leq \lambda \cdot \beta_n^R(k)$ *for some real number λ and each $n \geq 0$.*

The inequality in Proposition (9.1.2.1) suggests the use of an exponential scale to measure the growth of Betti numbers over arbitrary local rings.

9.2. Curvature. The *curvature* of M over R is defined in [13] to be

$$\operatorname{curv}_R M = \limsup \sqrt[n]{\beta_n^R(M)}$$

9.2.1. Remark. The first three relations below follow directly from the definitions, while the last one results from Proposition (9.1.2.2):

(1) $\operatorname{proj\,dim}_R M < \infty \iff \operatorname{cx}_R M = 0 \iff \operatorname{curv}_R M = 0$.
(2) $\operatorname{proj\,dim}_R M = \infty \iff \operatorname{cx}_R M \geq 1 \iff \operatorname{curv}_R M \geq 1$.
(3) $\operatorname{cx}_R M < \infty \implies \operatorname{curv}_R M \leq 1$.
(4) $\operatorname{cx}_R M \leq \operatorname{cx}_R k \quad \text{and} \quad \operatorname{curv}_R M \leq \operatorname{curv}_R k < \infty$.

A main open problem on the asymptotic behavior of Betti numbers is whether the converse of (3) holds. Whenever an answer is available, it is a consequence of some more precise statement on the growth of Betti numbers.

9.2.2. Theorem (Avramov [10], [14]). *If $\mathfrak{p}$ is a prime ideal of R such that $R_\mathfrak{p}$ is not a complete intersection, then*

$$\beta_n^R(R/\mathfrak{p}) \geq \beta^n \quad \textit{for some} \quad \beta > 1 \quad \textit{and} \quad \textit{for all} \quad n \gg 0$$

The converse may fail: for $R = k[t_1, t_2, t_3]/(t_1^2 t_3, t_2^2 t_3)$ and $\mathfrak{p} = (t_1, t_2)$ the ring $R_\mathfrak{p}$ is a complete intersection, but $\operatorname{curv}_R(R/\mathfrak{p}) > 1$.

An immediate application of the theorem is a characterization of complete intersections in terms of curvature, paralleling the one in terms of complexity in Proposition (6.2.1): that proposition provides the implication (i) $\implies$ (ii) below, while the theorem yields (iii) $\implies$ (i).

9.2.3. Corollary. *The following conditions are equivalent:*

(i) *R is complete intersection.*
(ii) $\operatorname{curv}_R N \leq 1$ *for every finite R-module N.*
(iii) $\operatorname{curv}_R k \leq 1$. □

Over some classes of rings, the Betti numbers of all modules are known to satisfy the polynomial/exponential dichotomy.

9.2.4. Theorem (Avramov [12], Sun [67]). *Set $c = \operatorname{edim} R - \operatorname{depth} R$.*
If $c \leq 3$, or if R is Gorenstein and $c \leq 4$, then either $\operatorname{cx}_R M \leq c$, or

$$\beta_n^R(M) \geq \beta \cdot \beta_{n-1}^R(M) \quad \textit{for some} \quad \beta > 1 \quad \textit{and} \quad \textit{for all} \quad n \gg 0$$

9.2.5. Theorem (Peeva [55]). *If $\operatorname{length}_R R \leq 8$, or if R is Gorenstein and $\operatorname{length}_R R \leq 11$, then either $\beta_n^R(M)$ is eventually constant, or*

$$\beta_n^R(M) \geq \beta \cdot \beta_{n-1}^R(M) \quad \textit{for some} \quad \beta > 1 \quad \textit{and} \quad \textit{for all} \quad n \gg 0$$

9.3. Extremal modules. Motivated by Proposition (9.1.2.2), we say that a module L over a local ring $(R, \mathfrak{m}, k)$ is *extremal* if $\operatorname{cx}_R L = \operatorname{cx}_R k$ and $\operatorname{curv}_R L = \operatorname{curv}_R k$. By the next theorem, such modules are plentyful.

9.3.1. Theorem (Avramov [13]). *If $i \geq 1$ and $\mathfrak{m}^i M \neq 0$, then the module $\mathfrak{m}^i M$ is extremal.*

The theorem is a quantitative sharpening of Theorem (1.2.2). It also implies that the Grothendieck group of R is generated by extremal R-modules:

9.3.2. Corollary. *Set $l = \operatorname{length}_R M$. If $l = 1$, then M is extremal. If $l > 1$, then there is an exact sequence $0 \to L \to M \to N \to 0$ with extremal L, N.*

Proof. Assume first $\mathfrak{m}M = 0$, so that $M \cong k^l$; if $l = 1$ then M is extremal by definition; if $l > 1$ then there is an exact sequence $0 \to k \to M \to k^{l-1} \to 0$ with the desired properties. Assuming $\mathfrak{m}M \neq 0$, the theorem shows that $\mathfrak{m}M$ is extremal, and the preceding discussion shows that so is $M/\mathfrak{m}M$. □

9.4. Remark. The proofs of many results in this section hinge on obtaining nontrivial information on the algebra $\operatorname{Ext}_R^{\bullet}(k,k)$ and on its action on $\operatorname{Ext}_R^{\bullet}(M,k)$, cf. (A.3). The cohomology algebra is itself the universal enveloping algebra of a functorial in R graded Lie algebra $\pi^{\bullet}(R)$, called the *homotopy Lie algebra* of R. It shares many properties with the graded Lie algebra of rational homotopy groups of a finite CW complex. This analogy is described in the papers [10], [18], along with applications of rational homotopy to local algebra, and *vice versa*.

Expositions. The survey [14] contains proofs of the results above.

10. Castelnuovo-Mumford regularity

We say that an algebra R over a field k is *graded* if $R = \bigoplus_{n \in \mathbb{N}} R_n$ as a k-vector space, $R_h \cdot R_i \subseteq R_{h+i}$ for all h, i, and $R_0 = k$.

10.1. Graded modules. An R-module M is *graded* if $M = \bigoplus_{j \in \mathbb{Z}} M_j$ with $R_i \cdot M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. For each $s \in \mathbb{Z}$, form the graded submodule $M_{\geqslant s} = \bigoplus_{j \geq s} M_j$ of M, and define a graded module $M(s)$ by setting $M(s)_j = M_{s+j}$ for all $j \in \mathbb{Z}$ and endowing it with the obvious action of R. Set

$$\inf M = \inf\{j \in \mathbb{Z} \mid M_j \neq 0\} \quad \text{and} \quad \sup M = \sup\{j \in \mathbb{Z} \mid M_j \neq 0\}$$

We say that M is *locally finite* if $\operatorname{rank}_k M_j$ is finite for all j. Abusing notation, we set $k = R/R_{\geqslant 1}$; up to shift, this is the only simple graded R-module.

Let $\mathcal{M}^{\mathrm{gr}}$ denote the category of graded R-modules with morphisms the R-linear maps $\beta\colon M \to N$ satisfying $\beta(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$.

It is easy to produce for every $M \in \mathcal{M}^{\mathrm{gr}}$ a *graded free resolution* $\boldsymbol{F} \to M$, with each F_i a direct sum of copies of $R(-j)$. In some cases, more is known.

10.2. Minimal resolutions. Let $M \in \mathcal{M}^{\mathrm{gr}}$ be a module with $\inf M > -\infty$.

10.2.1. Graded projective covers. Choose a k-linear section σ of the canonical projection $M \to M/R_{\geqslant 1}M$ and form the map

$$R \otimes_k (M/R_{\geqslant 1}M) \longrightarrow M \qquad \text{by setting} \qquad a \otimes x \longmapsto a\sigma(x)$$

It is easy to see that it is a projective cover of M in $\mathcal{M}^{\mathrm{gr}}$.

10.2.2. Minimal graded resolutions. Iterating the construction above, one obtains a free resolution $\boldsymbol{F} \to M$ in which $F_i \to \operatorname{Ker}(\partial_{i-1})$ is a graded projective cover for each i. By the graded version of Corollary (4.2.3) such a resolution is unique up to an isomorphism of complexes of graded R-modules.

It follows from the existence of minimal free resolutions that the global dimension of R is finite if and only if the projective dimension of k is finite. In the finitely generated commutative case, the algebras of finite global dimension are completely described by the next result, where (i) $\implies$ (ii) is *Hilbert's Syzygy Theorem*, while (iii) $\implies$ (i) is a consequence of Theorem (1.2.1).

10.2.3. Theorem. *If R is commutative and noetherian, then the following conditions are equivalent.*

(i) *R is a polynomial ring over k.*
(ii) $\operatorname{proj\,dim}_R N < \infty$ *for every finite graded R-module N.*
(iii) $\operatorname{proj\,dim}_R k < \infty$. □

The uniqueness of minimal free resolutions yields new numerical invariants.

10.2.4. Graded Betti numbers. The number $\beta_{ij}^R(M)$ of summands of F_i isomorphic to $R(-j)$ is an invariant of M, its ijth *graded Betti number*, so each graded R-module M module has a *Betti diagram* (not just a Betti sequence).

From the construction of graded projective covers one obtains equalities

$$\beta_{ij}^R(M) = 0 \qquad \text{for all} \qquad j < i + \inf M$$

and, if R and M are locally finite, inequalities $\beta_{ij}^R(M) < \infty$ for all $i, j \in \mathbb{Z}$.

If R is noetherian and M is finite, then every module in a minimal free resolution of M has finite rank, so $\beta_{ij}^R(M) = 0$ for all $j \gg 0$.

The next notion originates in algebraic geometry.

10.2.5. Castelnuovo-Mumford regularity. The number

$$\operatorname{reg}_R M = \sup\left\{n \in \mathbb{Z} \,\middle|\, \beta_{i,\,i+n}^R(M) \neq 0 \text{ for some } i \in \mathbb{N}\right\}$$

is known as the *Castelnuovo-Mumford regularity* of M over R.

In view of Hilbert's Syzygy Theorem and the last remark in (10.2.4), finite graded modules over a graded polynomial ring have finite regularity. This invariant plays important role in computational commutative algebra, cf. [34].

The next result shows some respects in which regularity behaves like a homological dimension, cf. Theorem (8.7.2) and Proposition (2.2.4).

10.2.6. Proposition. *There is an inequality* $\operatorname{reg}_R M \le \operatorname{reg}_R k \cdot \sup M$
If $0 \to M \to M' \to M'' \to 0$ *is an exact sequence in* $\mathcal{M}^{\mathrm{gr}}(R)$*, then*

$$\operatorname{reg}_R M \le \max\{\operatorname{reg}_R M',\, \operatorname{reg}_R M'' - 1\}$$

with equality if $\operatorname{reg}_R M' \ne \operatorname{reg}_R M''$.

Proof. The second assertion is obtained by standard use of mapping cones and the Horseshoe Lemma. For the first assertion, one may assume both $\operatorname{reg}_R k$ and $s = \sup M$ are finite, then induce on s by applying the already proven statement to the exact sequence $0 \to M_s(-s) \to M \to M/M_s \to 0$. □

10.3. Koszul algebras. The algebra R is said to be *Koszul* if $\operatorname{reg}_R k = 0$.

This is a simplification, widely adopted in texts on commutative algebra and algebraic geometry, of a notion introduced by Priddy [57]. Under this definition, a Koszul algebra is generated in degree 1 and has a set of quadratic defining relations. However, the Koszul property is much subtler and cannot be inferred from the knowledge of any *finite* number of Betti numbers of k.

10.3.1. Example (Roos [62]). Let $t_1, \dots, t_6$ be commuting indeterminates of degree 1 over k. For each integer n form the graded algebra

$$R_{(n)} = \frac{k[t_1, t_2, t_3, t_4, t_5, t_6]}{\begin{matrix}\big(t_1t_3 + nt_3t_6 - t_4t_6\,,\ t_1t_4 + t_3t_6 + (n-2)t_4t_6\,, \\ t_1^2\,, t_2^2\,, t_3^2\,, t_4^2\,, t_5^2\,, t_6^2\,, t_1t_2\,, t_2t_3\,, t_3t_4\,, t_4t_5\,, t_5t_6 \big)\end{matrix}}$$

If k has characteristic 0 and $n \ge 3$, then there are equalities

$$\beta_{ij}^{R_{(n)}}(k) = 0 \quad \text{for all} \quad j \ne i < n \quad \text{and} \quad \beta_{n,n+1}^{R_{(n)}}(k) \ne 0$$

A graded resolution $\boldsymbol{F} \to M$ is *linear* if there is an integer s, such that for each $n \ge 0$ the free R-module F_n is generated in degree $n + s$. For instance, the Koszul property of R can be described by the condition that the residue field has a linear resolution. The following was conjectured by Kempf.

10.3.2. Theorem (Avramov-Eisenbud [16]). *If R is commutative, noetherian, and Koszul, then for every finite R-module N there exists an integer m such that for each $s \ge m$ the module $N_{\geqslant s}$ has a linear resolution.*

It follows that M itself has finite regularity: the graded module $L = M_{\geqslant s}$ has $\operatorname{reg}_R L \le s$ by the theorem, so one obtains $\operatorname{reg}_R M \le s$ by applying Proposition (10.2.6) to the exact sequence $0 \to L \to M \to M/L \to 0$.

The last observation is generalized below.

10.4. Finite regularity. Let R be a commutative graded algebra.

The proof of the next result uses a graded version of Proposition (5.3.4).

10.4.1. Theorem (Avramov-Peeva [20]). *If R is noetherian and* $\operatorname{reg}_R k < \infty$, *then* $\operatorname{reg}_R N < \infty$ *for every finite graded module N.*

The next theorem, which does not require finite generation of R over k, pinpoints the relation of the Koszul property of R and the finiteness of $\operatorname{reg}_R k$.

10.4.2. Theorem (Avramov-Peeva [20]). *If* $\operatorname{reg}_R k < \infty$, *then* $S = k[R_1]$ *is a Koszul algebra and* R *is a polynomial ring in finitely many indeterminates over* S.

Conversely, if R *is a polynomial ring in indeterminates* $t_1, \dots, t_\ell$ *over a Koszul algebra, then* $\operatorname{reg}_R k = \sum_{p=1}^{\ell}(\deg(t_p) - 1)$.

Combining the theorems above, we obtain a complete characterization of graded algebras over which all modules have finite regularity.

10.4.3. Corollary. *If* R *is finitely generated as a* k*-algebra, then the following conditions are equivalent.*

(i) R *is a polynomial ring over a Koszul algebra.*
(ii) $\operatorname{reg}_R N < \infty$ *for every finite* R*-module* N.
(iii) $\operatorname{reg}_R k < \infty$. □

The form of the corollary brings to mind several results in this survey.

Appendix A. Complexes

Let R denote an associative ring, assumed to act on its modules from the left; right R-modules are treated as modules over the opposite ring, R^{o}.

A.1. Categories. We write $\mathcal{M}$ or $\mathcal{M}(R)$ for the category of R-modules. We only consider full additive subcategories of $\mathcal{M}$, closed under isomorphisms.

For any module M, left or right, the dual $M^* = \operatorname{Hom}_R(M, R)$ is given the canonical action on the other side. We let $(\ )^*$ denote either functor

$$\mathcal{M}(R)^{\mathrm{op}} \underset{\operatorname{Hom}_{R^{\mathrm{o}}}(\ ,R)}{\overset{\operatorname{Hom}_R(\ ,R)}{\rightleftarrows}} \mathcal{M}(R^{\mathrm{o}})^{\mathrm{op}}$$

A.2. Complexes. A *complex* of R-modules is a sequence of R-linear maps

$$\boldsymbol{B} = \quad \cdots \longrightarrow B_{n+1} \xrightarrow{\partial^{\boldsymbol{B}}_{n+1}} B_n \xrightarrow{\partial^{\boldsymbol{B}}_{n}} B_{n-1} \longrightarrow \cdots$$

with $\partial^{\boldsymbol{B}}_n \circ \partial^{\boldsymbol{B}}_{n+1} = 0$ for all n. We call $\Omega^n \boldsymbol{B} = \operatorname{Coker} \partial^{\boldsymbol{B}}_{n+1}$ the nth *syzygy module* of $\boldsymbol{B}$. Whenever convenient, we identify a module M with the complex having $M_0 = M$ and $M_n = 0$ for $n \neq 0$.

Every complex $\boldsymbol{B}$ of R-modules determines a complex of R^{o}-modules

$$\boldsymbol{B}^* = \quad \cdots \longrightarrow (B_{-n-1})^* \xrightarrow{(\partial^{\boldsymbol{B}}_{-n})^*} (B_{-n})^* \xrightarrow{(\partial^{\boldsymbol{B}}_{-n+1})^*} (B^*_{-n+1})^* \longrightarrow \cdots$$

where the module $(B_{-n})^*$ appears in degree n.

A *morphism* $\beta \colon \boldsymbol{B} \to \boldsymbol{C}$ of complexes is a sequence of R-linear maps $\beta_n \colon B_n \to C_n$ with $\partial^{\boldsymbol{C}}_n \beta_n = \beta_{n-1}\partial^{\boldsymbol{B}}_n$ for all $n \in \mathbb{Z}$. It is a *quasiisomorphism* if the induced map $\mathrm{H}_n(\beta) \colon \mathrm{H}_n(\boldsymbol{B}) \to \mathrm{H}_n(\boldsymbol{C})$ is bijective for every n.

If $\beta, \beta' \colon \boldsymbol{B} \to \boldsymbol{C}$ are morphisms, then a *homotopy* from β to β' is a sequence θ of R-linear maps $\theta_n \colon B_n \to C_{n+1}$ such that $\beta'_n = \beta_n + \partial^{\boldsymbol{C}}_{n+1}\theta_n + \theta_{n-1}\partial^{\boldsymbol{B}}_n$ for all $n \in \mathbb{Z}$. If such a homotopy exists, then β' is said to be *homotopic* to β, denoted $\beta' \sim \beta$; when this is the case, $\mathrm{H}_n(\beta) = \mathrm{H}_n(\beta')$ for all $n \in \mathbb{Z}$.

A complex $\boldsymbol{A}$ is said to be *contractible* (or *split exact*) if $\mathrm{id}_{\boldsymbol{A}} \sim 0_{\boldsymbol{A}}$.

A morphism β is a *homotopy equivalence* if it has a *homotopy inverse*, that is, a morphism γ such that $\beta\gamma \sim \mathrm{id}_{\boldsymbol{C}}$ and $\gamma\beta \sim \mathrm{id}_{\boldsymbol{B}}$. The following implications, non-invertible in general, obviously hold for every morphism:

$$\text{isomorphism} \implies \text{homotopy equivalence} \implies \text{quasiisomorphism}$$

Let M be an R-moduled and let $\mathcal{C}$ be a subcategory of $\mathcal{M}(R)$. A *$\mathcal{C}$-resolution* (respectively, *$\mathcal{C}$-coresolution*) of M is a quasiisomorphism $\gamma\colon \boldsymbol{C} \to M$ (respectively, $M \to \boldsymbol{C}$) where $\boldsymbol{C}$ a complex $\boldsymbol{C}$ with $C_n \in \mathcal{C}$ for all $n \in \mathbb{Z}$ and $C_n = 0$ for all $n < 0$ (respectively, for all $n > 0$).

A.3. Products in (co)homology. Let L, M, N be R-modules.

Comparison morphisms for projective resolutions provide natural maps

$$\operatorname{Ext}^i_R(M,N) \otimes_{\mathbb{Z}} \operatorname{Ext}^j_R(L,M) \longrightarrow \operatorname{Ext}^{i+j}_R(L,N)$$

These pairings are associative, and so give $\operatorname{Ext}^{\bullet}_R(N,N)$ $\operatorname{Ext}^{\bullet}_R(M,M)$ structures of graded R-algebras, and $\operatorname{Ext}^{\bullet}_R(M,N)$ a structure of bimodule with $\operatorname{Ext}^{\bullet}_R(N,N)$ acting from the left and $\operatorname{Ext}^{\bullet}_R(M,M)$ from the right.

Comparison morphisms also define natural homomorphisms of R-modules

$$\operatorname{Ext}^i_R(M,M) \otimes_{\mathbb{Z}} \operatorname{Tor}^R_j(M,N) \longrightarrow \operatorname{Tor}^R_{j-i}(M,N)$$
$$\operatorname{Tor}^R_j(M,N) \otimes_{\mathbb{Z}} \operatorname{Ext}^i_R(N,N) \longrightarrow \operatorname{Tor}^R_{j-i}(M,N)$$

that turn $\operatorname{Tor}^R_j(M,N)$ into a bimodule.

Appendix B. Local rings

In this Appendix $(R,\mathfrak{m},k)$ denotes a *local ring*, that is, R is a commutative noetherian ring, $\mathfrak{m}$ is its unique maximal ideal, and k is its residue field $R/\mathfrak{m}$. Furthermore, M denotes an R-module which is assumed to be *finite*, in the sense that it is finitely generated over R. We define some basic numerical invariants attached to M and R, and to use them to give concise definitions of the main classes of commutative local rings.

B.1. Depth. This concept extends the notion of non-invertible-non-zero-divisor: A sequence $\boldsymbol{y} = y_1, \dots, y_r$ of elements of R is *M-regular* if $\boldsymbol{y}M \neq M$, and for $i = 1, \dots, r$ multiplication with y_i gives an injective map

$$M/(y_1, \dots, y_{i-1})M \xrightarrow{y_i} M/(y_1, \dots, y_{i-1})M$$

Let I be an ideal of R. Due to the noetherian character of R, every M-regular sequence $\boldsymbol{y}$ contained in I can be extended to one which is *maximal* with respect to inclusion. The *depth of I on M* is the number

$$\operatorname{depth}_R(I,M) = \sup\{r \in \mathbb{N} \mid I \text{ contains an } M\text{-regular sequence of length } r\}$$

B.1.1. Theorem (Rees [58]). *Let I be an ideal of R, let L be some finite R-module with $\operatorname{rad}(\operatorname{ann}_R(L)) = \operatorname{rad}(I)$, and let $\boldsymbol{y}$ be a maximal M-regular sequence in I. If $\operatorname{depth}_R(I,M) = g$, then the following then hold.*

(1) *$\boldsymbol{y}$ contains g elements.*
(2) $g = \inf\{n \in \mathbb{N} \mid \operatorname{Ext}^n_R(L,M) \neq 0\}$.
(3) $\operatorname{Ext}^g_R(R/I,M) \cong \operatorname{Hom}_R(R/I, M/\boldsymbol{y}M)$.
(4) *Every sequence of elements of R minimally generating $(\boldsymbol{y})$ is M-regular.*

A useful way to compute the depth of the ideal generated by a subset $\boldsymbol{y} = \{y_1, \dots, y_r\} \subseteq R$ is by using the *Koszul complex* $\mathrm{K}(\boldsymbol{y};M)$. It is concentrated in degrees 0 through r and may be defined in two steps as follows:

$$\mathrm{K}(y_1;R) = \ 0 \longrightarrow R \xrightarrow{y_1} R \longrightarrow 0 \quad \text{and} \quad \mathrm{K}(\boldsymbol{y};R) = \bigotimes_{i=1}^{r} \mathrm{K}(y_i;R)$$

The next result is obvious for $r = 1$; the general case is obtained by induction.

B.1.2. Theorem (Auslander-Buchsbaum [6]). *If* $\boldsymbol{y} = \{y_1, \dots, y_r\}$, *then*

$$\operatorname{depth}_R((\boldsymbol{y}), M) = r - \sup\{n \in \mathbb{Z} \mid \mathrm{H}_n\left(\mathrm{K}(\boldsymbol{y}; R) \otimes_R M\right) \neq 0\}$$

The number $\operatorname{depth}_R(\mathfrak{m}, M)$ is called the *depth* of M over R, denoted $\operatorname{depth}_R M$ (the subscript is dropped if $M = R$). By the convention $\sup\{\varnothing\} = -\infty$ and Nakayama's Lemma, $\operatorname{depth}_R M = -\infty$ if and only if $M = 0$.

B.1.3. Grade. The *grade* of M over R, denoted $\operatorname{grade}_R M$, is defined to be the number $\operatorname{grade}_R M = \operatorname{depth}_R(\operatorname{ann}_R M, R)$ if $\operatorname{ann}_R M = 0$, and to be 0 otherwise; we set $M_R^\dagger = \operatorname{Ext}_R^g(M, R)$, where $g = \operatorname{grade}_R M$.

By Theorem (B.1.1.2), if $M \neq$ then $\operatorname{grade}_R M \leq \operatorname{proj\,dim}_R M$. If equality holds, then M is said to be *perfect*. A $\mathcal{P}$-resolution $\boldsymbol{P} \to M$ of length $g = \operatorname{proj\,dim}_R M$ then yields $\operatorname{Ext}_R^n(M, R) = \mathrm{H}^n \operatorname{Hom}_R(\boldsymbol{P}, R) = 0$ for $n \neq g$. It follows that the R-module $M_R^\dagger = \mathrm{H}^g \operatorname{Hom}_R(\boldsymbol{P}, R)$ is perfect, has $\operatorname{proj\,dim}_R(M_R^\dagger) = g$, and satisfies $M_{RR}^{\dagger\dagger} \cong M$.

B.2. Hierarchy of local rings. Let $(R, \mathfrak{m}, k)$ be a local ring.

If there exists an R-regular sequence $\boldsymbol{y}$ such that the residue ring $R/(\boldsymbol{y})$

- is a field, then R is called *regular*.
- is isomorphic to $R'/(\boldsymbol{y}')$ for some regular ring R' and some R'-regular sequence $\boldsymbol{y}'$, then R is called *complete intersection*.
- has finite length and $\operatorname{Hom}_R(k, R/(\boldsymbol{y})) \cong k$, then R is called *Gorenstein*.
- has finite length, then R is called *Cohen-Macaulay*.

These properties can be introduced in several equivalent ways. Our choice of definitions is tailored to the needs of the survey.

B.2.1. Remark. The following implications hold:

$$\text{regular} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay}$$

To verify the first implication set $R' = R$ and $\boldsymbol{y}' = \varnothing$. For the middle one, note the isomorphisms $\operatorname{Hom}_R(k, R/(\boldsymbol{y})) \cong \operatorname{Hom}_{R'}(k, R'/(\boldsymbol{y}')) \cong \operatorname{Hom}_{R'}(k, k) \cong k$, where the second one is given by Theorem (B.1.1.3) because the maximal ideal $\mathfrak{m}'$ of R' is generated by an R'-regular sequence. The last implication holds tautologically.

Local rings often appear as homomorphic images of regular local rings. In algebraic geometry, these are localizations of polynomial rings over a field. In analytic geometry, these are rings of convergent power series (usually over $\mathbb{C}$ or $\mathbb{R}$). Theorem (B.4.3) yields a third source of such presentations. It is therefore important to be able to recognize when a residue ring of a regular ring belongs to one of the basic classes.

B.2.2. Theorem. *If* $(Q, \mathfrak{n}, k)$ *is a regular local ring,* $J \subset Q$ *an ideal,* $\boldsymbol{x}$ *a minimal set of generators of* J *and* $R = Q/J$, *then the following hold.*

1. R *is* regular *if and only if* $\boldsymbol{x}$ *is linearly independent modulo* $\mathfrak{n}^2$.
2. R *is* complete intersection *if and only if* $\boldsymbol{x}$ *is a regular sequence.*
3. R *is* Gorenstein *if and only if the* Q*-module* R *is perfect and* $R_Q^\dagger \cong R$.
4. R *is* Cohen-Macaulay *if and only if the* Q*-module* R *is perfect.*

B.3. Dimension. The primary measure of the size of M is its (Krull) *dimension* $\dim_R M$, equal to the least $d \in \mathbb{N}$ for which there is a set $\boldsymbol{b} = \{b_1, \dots, b_d\} \subseteq \mathfrak{m}$, such that the R-module $M/\boldsymbol{b}M$ has finite length. One sets $\dim R = \dim_R R$. Note that $\dim R = 0$ if and only if R is artinian.

The cornerstone of dimension theory is *Krull's Principal Ideal Theorem*. Here is a generalization for modules:

B.3.1. Theorem. *If M has finite length and $R^s \to R^r \to M \to 0$ is an exact sequence, then* $\dim R = 0$ *or* $\dim R \leq s - r + 1$.

Inequalities link depth, dimension, and the number $\operatorname{edim} R = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, called the *embedding dimensionnequalities* of R; the last one comes by applying Theorem (B.3.1) to an exact sequence $R^s \to R \to k \to 0$ with $s = \operatorname{edim} R$.

B.3.2. Theorem. $\operatorname{depth}_R M \leq \dim_R M \leq \dim R \leq \operatorname{edim} R$.

We can now give standard definitions of two of the concepts defined in (B.2); for the other two, cf. Remark (B.4.4) and Theorem (1.4.1).

B.3.3. Remark. R is regular if and only if $\dim R = \operatorname{edim} R$.
R is Cohen-Macaulay if and only if $\dim R = \operatorname{depth} R$.

B.4. Completion. The $\mathfrak{m}$-*adic topology* is the linear topology in which a basis of neighborhoods of 0 is given by the family of ideals $\{\mathfrak{m}^n\}_{n\geqslant 0}$. The *completion* $\widehat{R}$, as usual, consists of equivalence classes of Cauchy sequences, and the *completion map* $R \to \widehat{R}$ sends each element to the class of the constant sequence it determines.

B.4.1. Theorem. *$\widehat{R}$ is a local ring with maximal ideal $\mathfrak{m}\widehat{R}$ and residue field k; its depth, dimension, and embedding dimension are equal to those of R.*

The completion map is a faithfully flat homomorphism, and so injective.

The ring $\widehat{R}$ is regular (respectively, complete intersection, Gorenstein, Cohen-Macaulay) if and only R has the corresponding property.

The completion map is bijective if and only if R is *complete*.

B.4.2. Examples. If R is artinian, then $\mathfrak{m}$ is nilpotent, so the $\mathfrak{m}$-adic topology is discrete, hence R is trivially complete. If R is complete, then so is the ring $R[[\boldsymbol{t}]]$ of formal power series in any finite set of indeterminates $\boldsymbol{t}$. For every R, the completion of $R[\boldsymbol{t}]_{(\mathfrak{m},\boldsymbol{t})}$ is isomorphic to $\widehat{R}[[\boldsymbol{t}]]$.

A local ring is called *equicharacteristic* if it is an algebra over some field.

B.4.3. Theorem (Cohen [30]). *If $(R, \mathfrak{m}, k)$ is any local ring, then $\widehat{R} \cong Q/J$ for some complete regular local ring $(Q, \mathfrak{n}, k)$, which can be chosen to be a ring of formal power series over the field k if R is equicharacteristic, and over a complete discrete valuation ring with residue field k otherwise.*

Any isomorphism $\widehat{R} \cong Q/J$ where Q ia a complete regular local ring is called a *Cohen presentation* of $\widehat{R}$. There always exists a Cohen presentation $\widehat{R} \cong Q/J$ which is *minimal*, in the sense that $\operatorname{edim} Q = \operatorname{edim} R$.

As an application of Cohen's Theorem, we give the standard definition of the complete intersection property.

B.4.4. Remark. If R is complete intersection, then in every Cohen presentation $\widehat{R} \cong Q/J$ the ideal J can be generated by a regular sequence. Conversely, if in some presentation $\widehat{R} \cong Q/J$ the ideal J is generated by a regular sequence, then R is a complete intersection.

Indeed, by Theorem (B.4.1), R is complete intersection if and only if so is $\widehat{R}$. By Theorem (B.2.2.2), $\widehat{R}$ is complete intersection if and only if J is generated by a regular sequence $\boldsymbol{x} = x_1, \ldots, x_c$.

Expositions. Modern accounts of commutative ring theory are built around the concepts discussed above. The books of Matsumura [52], Bruns and Herzog [26], and Eisenbud [35] provide complementary approaches to the material.

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