COHEN–MACAULAY PROPERTIES OF RING HOMOMORPHISMS

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Abstract. Numerical invariants which measure the Cohen–Macaulay character of homomorphisms $\varphi\colon R \to S$ of noetherian rings are introduced and studied. Comprehensive results are obtained for homomorphisms which are locally of finite flat dimension. They provide a point of view from which a variety of phenomena receive a unified treatment. The conceptual clarification and technical versatility of this approach leads, among other things, to a determination of those homomorphisms which preserve the Cohen–Macaulay character of the rings, to the discovery of new classes of homomorphisms with remarkable stability properties, and to solutions of some problems on flat homomorphisms, raised by Grothendieck in EGA.

Contents

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Introduction

A systematic emphasis on the study of morphisms, rather than objects, was an innovative[1] aspect of Grothendieck's approach to algebraic geometry and commutative algebra. A detailed account of *local properties* of locally noetherian schemes and their morphisms is the main subject of Chapter IV of EGA [16]. There, a morphism is said to have some local property, like being regular, normal, Cohen–Macaulay, of codepth at most n, etc., if it is *flat*, and its non-trivial fibers have the geometric form of the corresponding property.

Powerful as it has proved to be, this point of view has some undesirable limitations. On the one hand, "relative" properties of morphisms do not incorporate "absolute" properties of schemes, except when the latter are defined over some field. Most notably, local properties of $\operatorname{Spec} S$ when S is a ring are not equivalent to the corresponding properties of the structural morphism $\operatorname{Spec} S \to \operatorname{Spec} \mathbb{Z}$. On the other hand, useful results on the stability of local properties under important types of morphisms – e.g. regular embeddings – find no explanation within this framework.

The common feature of flat morphisms, structural morphisms, and regular embeddings, is that locally they are of *finite flat dimension* (also known as Tor-dimension). In dealing with such maps one is led to consider homomorphisms $R \to S$ of commutative noetherian rings, such that for each $\mathfrak{q} \in \operatorname{Spec} S$ the R–module $S_{\mathfrak{q}}$ has a finite resolution by flat modules.

In [6] some classical numerical measures of "size" and "quality" of rings have been extended to homomorphisms. For properties related to being locally complete intersection, Gorenstein, or Cohen–Macaulay, this opens a new perspective on the study of both objects and maps. In particular, it extends techniques available earlier only in the flat case to a wide class of homomorphisms, which includes all those locally of finite flat dimension.

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In this context, many fundamental relations take the concise form of (in)equalities between numerical invariants of local homomorphisms. To explain the last fact we recall that by a basic tenet of modern commutative algebra the size of a noetherian ring is measured by various numerical parameters, and simple relations between them have strong structural implications. For example, there is an inequality $\dim R \geq \operatorname{depth} R$ between the dimension of a local ring $(R, \mathfrak{m})$ and its depth, and equality singles out the remarkable class of Cohen–Macaulay rings. Thus, the *Cohen–Macaulay defect* $\operatorname{cmd} R = \dim R - \operatorname{depth} R$ is a non-negative integer which determines the failure of R to be Cohen–Macaulay (in EGA it is called the *codepth* of R and is denoted $\operatorname{coprof} R$).

Various qualitative results on Cohen–Macaulay rings are in fact limit cases of quantitative relations involving this invariant. For instance, the passage of the Cohen–Macaulay property to localizations and formal fibers is contained in the inequality

$$\operatorname{cmd} R_{\mathfrak{q}} + \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)_{\widetilde{\mathfrak{p}}} \leq \operatorname{cmd} R\,,$$

[1] To quote from Serre's 1965 description of Grothendieck's 1957 proof of the Riemann–Roch Theorem: "A côté de difficultés techniques considérables, elle contient deux idées, totalement originales, qui ont eu une grande influence: (a) La plupart des notions, ou théorèmes, relatifs aux variétés, doivent être étendus aux *morphismes*. Ainsi, pour prendre des exemples élémentaires, la notion d'espace compact correspond à celle d'application propre; celle de variété non singulière à celle de morphisme lisse; etc.", cf. [34, p. 201].

where $\mathfrak{p} \in \operatorname{Spec} R$, $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, and $\widetilde{\mathfrak{p}}$ is an arbitrary prime ideal in $k(\mathfrak{p}) \otimes_R \widehat{R}$. In particular, this shows that a Cohen–Macaulay defect may be introduced for any noetherian ring R by setting $\operatorname{cmd} R = \max \{ \operatorname{cmd} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \}$.

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This paper analyzes Cohen–Macaulay properties of homomorphisms of noetherian rings. As an illustration of the approach taken here we describe how it applies to a long standing Localization Problem, considered by Grothendieck in [16, (7.5.4)]:

> Let $\varphi\colon R \to S$ be a flat local homomorphism of local rings, and assume that for each $\mathfrak{p} \in \operatorname{Spec} R$ the formal fiber $k(\mathfrak{p}) \otimes_R \widehat{R}$ is Cohen–Macaulay. If the closed fiber $S/\mathfrak{m}S$ of φ at the maximal ideal $\mathfrak{m}$ of R is Cohen–Macaulay, then does each fiber $k(\mathfrak{p}) \otimes_R S$ of φ have the same property?

We prove a result which goes beyond the original setup in three independent directions: It imposes no conditions on the ring R, removes the assumption on the closed fiber of φ, and relaxes the condition on the homomorphism from being flat to having finite flat dimension.

To do this, we assign to any local homomorphism $\varphi\colon R \to S$ an integer $\operatorname{cmd} \varphi$, called its Cohen–Macaulay defect, and show that $\operatorname{cmd} \varphi = \operatorname{cmd} S$ when R is Cohen–Macaulay, that $\operatorname{cmd} \varphi \geq 0$ when the R–module S has finite flat dimension, and that $\operatorname{cmd} \varphi = \operatorname{cmd}(S/\mathfrak{m}S)$ when it is flat. Thus, a solution of the Localization Problem is contained in the inequality

$$\operatorname{cmd} \varphi_{\mathfrak{q}} + \operatorname{cmd}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big) \leq \operatorname{cmd} \varphi + \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)\,,$$

which holds when φ has finite flat dimension, $\mathfrak{q}$ is an arbitrary prime ideal in S, $\mathfrak{p} = \mathfrak{q} \cap R$, and $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is the induced local homomorphism.

A self-contained proof of the inequality for flat φ is presented in [4]. This also solves a more general problem of Grothendieck [*idem.*]: if R has Cohen–Macaulay formal fibers and the closed fiber of φ has Cohen–Macaulay defect at most n, then so do all its fibers. The reader might consult [4] for a discussion of earlier and related results. Here we only note that even when φ is flat, our proofs use the methods of the present paper, and may not be carried out within the framework of flat homomorphisms.

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The study of Cohen–Macaulay defects of local homomorphisms and of their extensions to homomorphisms of noetherian rings forms the substance of this paper. Our approach to local properties of maps is based on a structure theorem [6] for a local homomorphism to a complete local ring. It shows that such a homomorphism decomposes into a faithfully flat inclusion with regular closed fiber followed by a surjection: this represents a relative version of Cohen's Structure Theorem for complete local rings.

The construction of such a *Cohen factorization* is used in the definition [6] of the Cohen–Macaulay defect of a local homomorphism $\varphi\colon R \to S$, which we recall in Section 3. The rest of that section (re)establishes, sometimes with new proofs, some of the more elementary properties of $\operatorname{cmd}\varphi$. It also introduces the useful homological formula

$$\operatorname{cmd}\varphi = \operatorname{cmd}_S\big(D \overset{\mathbf{L}}{\otimes}_R S\big)\,,$$

where D is a dualizing complex for R, while tensor product and Cohen–Macaulay defect are computed in the appropriate derived categories.

The basic relations between the Cohen–Macaulay defects of φ, a second local homomorphism $\psi\colon Q \to R$, and their composition appear in the form of an inequality

$$\operatorname{cmd}\varphi\psi \le \operatorname{cmd}\psi + \operatorname{cmd}\varphi\,,$$

which becomes an equality when ψ has finite flat dimension, and φ is flat or R is Cohen–Macaulay. It is not known whether equality holds when both ψ and φ have finite flat dimension, and in an Appendix we show that such an equality is equivalent to a well known conjecture of M. Auslander: The grade of an ideal $\mathfrak{a}$ of finite projective dimension in a local ring R is equal to its codimension $\dim R - \dim R/\mathfrak{a}$. As a partial result in this direction, we prove that there are inequalities

$$\operatorname{cmd}\psi \le \operatorname{cmd}\varphi\psi \quad \text{and} \quad \operatorname{cmd}\varphi \le \operatorname{cmd}\varphi\psi\,.$$

The proofs of the results on composition involve a variety of techniques. In Section 2 we compare the codimensions in $\operatorname{Spec} R$ and in $\operatorname{Spec} Q$ of the support a finite R–module M, when R is a homomorphic image of a local ring Q. This analysis uses computations in the derived category, which draw heavily on Iversen's Amplitude Inequality [21] and on the New Intersection Theorem of Peskine and Szpiro [28], Hochster [19], and P. Roberts [30], [32]. The relations above are established in Section 4, based on the results of the preceding sections and on a construction of Cohen factorizations for a composition of homomorphisms, which is useful in other situations as well.

The general solution of Grothendieck's Localization Problem, discussed above, is given in Section 5. In Section 6 we investigate two globalizations of the notion of Cohen–Macaulay defect to homomorphisms $\varphi\colon R \to S$ of noetherian rings, bound their difference in terms of invariants of the formal fibers of R, and establish various stability properties under flat base change.

In Section 7, by using once more Cohen factorizations, we introduce for local homomorphisms a notion of type which extends one of those available for local rings. The basic properties of this invariant are essentially multiplicative forms of those of the Cohen–Macaulay defect, with an important simplification: there is an equality

$$\operatorname{type}\varphi\psi = \operatorname{type}\psi \cdot \operatorname{type}\varphi$$

whenever φ has finite flat dimension, and ψ has finite flat dimension or Q is Gorenstein.

The final Section 8 studies the relative version of the Cohen–Macaulay property. A homomorphism of noetherian rings is said to be *locally Cohen–Macaulay* if it is locally of finite flat dimension, and $\operatorname{cmd}\varphi_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \operatorname{Spec} S$. Cohen–Macaulay homomorphisms are further stratified by type. The thinnest stratum – consisting of those of type 1 – is formed by the *locally Gorenstein homomorphisms*, introduced and studied in [1] and [3] from a completely different point of view.

While the theorems listed in the last section are mostly obtained as specializations of earlier results, the reader might gain some insight into the working of the present theory by browsing through them, and spelling out their specific forms for structure homomorphisms, flat homomorphisms, or factorizations of regular sequences. Typically, *structural* results for Cohen–Macaulay rings are recovered as special cases of *stability* results for Cohen–Macaulay homomorphisms.

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The rings considered below are commutative and noetherian. Even when not expressed, this condition is present in the statements of all results.

1. Commutative algebra of complexes

We fix a local ring $(R, \mathfrak{m}, k)$, that is, a noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. This short section puts together a minimum of information pertaining to the derived category of R–modules.

Complexes of R–modules are graded by subscripts and have differentials of degree -1. If N is an R–module then we also denote by N the complex of R–modules with $N_i = 0$ for $i \neq 0$ and $N_0 = N$. Conversely, if a complex M has $M_i = 0$ for $i \neq 0$ we identify M with the module M_0 and say that M is an R–module.

Morphisms of complexes are R–linear maps of degree zero which commute with the differentials. A morphism which induces an isomorphism in homology is called a *quasi-isomorphism*. We use the symbol $\simeq$ to indicate quasi-isomorphisms, while $\cong$ is our notation for isomorphisms of complexes (and thereby of modules). The derived category $\mathbf{D}(R)$ of R–modules is the localization of the category of complexes of R–modules at the class of all quasi-isomorphisms, cf. [36], [17], and also [15] and [14] for more detailed accounts. We use the notation $\simeq$ for isomorphisms in the derived category: note that a morphism of complexes is a quasi-isomorphism precisely when it represents an isomorphism in the derived category.

The *infimum*, *supremum*, and *amplitude* of a complex M are defined by

$$\begin{aligned} \inf M &= \inf \left\{ i \in \mathbb{Z} \mid \mathrm{H}_i(M) \neq 0 \right\}; \\ \sup M &= \sup \left\{ i \in \mathbb{Z} \mid \mathrm{H}_i(M) \neq 0 \right\}; \\ \operatorname{amp} M &= \sup M - \inf M\,, \end{aligned}$$

when $\mathrm{H}(M) \neq 0$; otherwise we set $\inf M = \infty$ and $\sup M = \operatorname{amp} M = -\infty$. When $\operatorname{amp} M$ is finite and the R–module $\mathrm{H}_i(M)$ is finite for each $i \in \mathbb{Z}$, we say that $\mathrm{H}(M)$ is *finite*.

A complex M of R–modules with $\operatorname{amp} M < \infty$ is said to have *finite projective* (respectively, *injective*) *dimension* if $M \simeq X$ with X a complex of projective (respectively, injective) R–modules such that $X_i = 0$ for $|i| \gg 0$. When this is the case, we write $\operatorname{pd}_R M < \infty$ (respectively, $\operatorname{id}_R M < \infty$).

We use the right derived functor of the homomorphism functor of complexes of R–modules, denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$, and the left derived functor of the tensor product functor of complexes of R–modules, denoted by $-\overset{\mathbf{L}}{\otimes}_R-$. Thus, $\mathbf{R}\mathrm{Hom}_R(M,N)$ and $M\overset{\mathbf{L}}{\otimes}_R N$ are complexes which are defined uniquely up to isomorphism in $\mathbf{D}(R)$, and have the usual functorial properties. We impose no boundedness conditions on the complex arguments, due to the existence of appropriate resolutions, established in [35]. As usual, for $i \in \mathbb{Z}$ we set $\operatorname{Ext}^i_R(M,N) = \mathrm{H}_{-i}\big(\mathbf{R}\mathrm{Hom}_R(M,N)\big)$ and $\operatorname{Tor}^R_i(M,N) = \mathrm{H}_i(M \overset{\mathbf{L}}{\otimes}_R N)$; these are the classical notions when M and N are modules.

(1.1) *Numerical invariants.* Familiar invariants of R–modules have been extended to complexes in several non-equivalent ways. We use the notions introduced in [12].

The (*Krull*) *dimension* of a complex M of R–modules is defined in terms of the (Krull) dimensions of its homology modules by the formula:

$$\dim_R M = \sup\left\{ \dim_R \mathrm{H}_i(M) - i \mid i \in \mathbb{Z} \right\},$$

with the convention that the dimension of the zero module is equal to $-\infty$. The *depth* of a complex M of R–modules is defined by the formula

$$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k,M)\,,$$

hence $-\infty \leq \operatorname{depth}_R M \leq \infty$. In case M is an R–module the notions of dimension and depth coincide with the standard ones.

Extending the notation used for modules, for a complex M we set

$$\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M,$$

and call this number the *Cohen–Macaulay defect* of M.

(1.2) *Flat base change.* Numerical invariants of modules are known to behave nicely under flat base change, and this extends to the corresponding invariants of complexes.

Proposition. *Let $\varphi : (R,\mathfrak{m},k) \to (S,\mathfrak{n},\ell)$ be a flat homomorphism of local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. For a complex M of R–modules with $\mathrm{H}(M)$ finite there are equalities*

$$\begin{aligned}
\dim_S(M \otimes_R S) &= \dim_R M + \dim(S/\mathfrak{m}S)\,.\\
\operatorname{depth}_S(M \otimes_R S) &= \operatorname{depth}_R M + \operatorname{depth}(S/\mathfrak{m}S)\,.\\
\operatorname{cmd}_S(M \otimes_R S) &= \operatorname{cmd}_R M + \operatorname{cmd}(S/\mathfrak{m}S)\,.
\end{aligned}$$

Proof. It is well known that $\dim_S(H \otimes_R S) = \dim_R H + \dim(S/\mathfrak{m}S)$ for finite R–modules H, cf. e.g. [23, (15.1.ii)]. The result for complexes follows from the definition (1.1) of dimension, and the isomorphisms $\mathrm{H}_i(M \otimes_R S) \cong \mathrm{H}_i(M) \otimes_R S$ for $i \in \mathbb{Z}$.

The equality of depths is a consequence of the isomorphism

$$\mathbf{R}\mathrm{Hom}_S(\ell, M \otimes_R S) \simeq \mathbf{R}\mathrm{Hom}_R(k, M) \otimes_k \mathbf{R}\mathrm{Hom}_{S/\mathfrak{m}S}(\ell, S/\mathfrak{m}S)\,, \tag{1.2.1}$$

proved in [13] for modules by an argument which generalizes to complexes; the equality is also a special case of [8, (5.3)], itself a consequence of the main theorem of [7] and [8].

The formula for Cohen–Macaulay defects follows from the other two. □

(1.3) *Dualizing complexes.* A complex C of R–modules is said to be *dualizing* for R if $\mathrm{H}(C)$ is finite, the canonical map $\eta_R^C\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathbf{D}(R)$, and $\mathrm{id}_R\, C < \infty$. Equivalently, C is dualizing if $\mathrm{H}(C)$ is finite and, up to a shift, $\mathbf{R}\mathrm{Hom}_R(k, C)$ is isomorphic to k in $\mathbf{D}(R)$, cf. [17, (V.3.4)].

For further reference we collect some properties of dualizing complexes.

(1.3.1) Any complete local ring has a dualizing complex, cf. [17, (V.10.4)].

(1.3.2) If M is a complex of R–modules with $\mathrm{H}(M)$ finite and C is a dualizing complex for R, then $\mathrm{amp}\,\mathbf{R}\mathrm{Hom}_R(M, C) = \mathrm{cmd}_R\, M$ by [12, (8.12)].

(1.3.3) If $\varphi\colon R \to S$ is a finite local homomorphism of local rings and C is a dualizing complex for R, then $\mathbf{R}\mathrm{Hom}_R(S, C)$ is a dualizing complex for S, cf. [17, (V.10.2)].

(1.3.4) Let $R \to S$ be a flat homomorphism of local rings with $S/\mathfrak{m}S$ a local Gorenstein ring; if C is a dualizing complex for R, then $C \otimes_R S$ is one for S, cf. (1.2.1).

2. Codimension and finite projective dimension

In this section R denotes a local ring. The *codimension* in $\mathrm{Spec}\, R$ of the support of a finite R–module M is defined to be the height of the annihilator of M:

$$\mathrm{codim}_R\, M = \mathrm{ht}\,\mathrm{Ann}_R\, M\,.$$

We analyze relations between the codimensions of M over R and over a local ring of which R is a homomorphic image. Best results are obtained in the presence of finite projective dimension. The remarkable properties of the supports of such modules have motivated a substantial body of research over the past 25 years; its beginning may be traced to the announcement in [26], under a long and evocative title, of parts of Peskine and Szpiro's thesis [27].

(2.1) **Theorem.** *Let R be a homomorphic image of a local ring Q. If $M \neq 0$ is a finite R–module, then*

$$\mathrm{codim}_R\, M + \mathrm{codim}_Q\, R \le \mathrm{codim}_Q\, M\,. \tag{a}$$

$$\mathrm{codim}_Q\, M \le \mathrm{codim}_R\, M + \mathrm{pd}_Q\, R\,. \tag{b}$$

If furthermore $\mathrm{pd}_Q\, R$ is finite, then

$$\mathrm{codim}_Q\, M \le \mathrm{pd}_R\, M + \mathrm{codim}_Q\, R\,. \tag{c}$$

(2.2) *Remarks.* (1) Of the preceding statements, (a) is straightforward, and (b) is a fairly direct consequence of the New Intersection Theorem; the proof of (c) uses this theorem, and computations with derived functors.

(2) We know of no example of strict inequality in (a) when the projective dimensions of the R–module M and the Q–module R are both *finite*. In the Appendix we show that under these conditions equality is equivalent to a well known conjecture of M. Auslander.

Proof of (2.1.a). Choose a prime $\mathfrak{q} \in \operatorname{Supp}_Q M$ with $\operatorname{codim}_Q M = \dim Q_{\mathfrak{q}}$ and set $\mathfrak{p} = \mathfrak{q}R \in \operatorname{Supp}_R M$. Using the isomorphism $R_{\mathfrak{q}} \cong R_{\mathfrak{p}}$ and the preceding inequalities, one gets

$$\operatorname{codim}_Q M = \dim Q_{\mathfrak{q}} \geq \dim R_{\mathfrak{p}} + \operatorname{codim}_{Q_{\mathfrak{q}}} R_{\mathfrak{q}} \geq \operatorname{codim}_R M + \operatorname{codim}_Q R\,. \qquad \square$$

(2.3) *Notions of codimension.* For a finite R–module $M \neq 0$ and for each $\mathfrak{p} \in \operatorname{Supp}_R M$ there are inequalities

$$\text{(2.3.1)} \qquad \operatorname{depth}_R M \leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim(R/\mathfrak{p}) \leq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim(R/\mathfrak{p}) \leq \dim_R M\,.$$

The non-negative differences implicit in this sequence control the gaps between $\operatorname{codim}_R M$ and its approximations by other invariants.

The maximal length of an R–regular sequence contained in the annihilator of M is called the *grade* of M and is denoted $\operatorname{grade}_R M$. As an inequality $\operatorname{grade}_R M \leq \operatorname{codim}_R M$ holds always, this invariant provides a natural lower bound for codimension. Furthermore, being of a homological nature, grade is easier to track through changes of rings. Often it is convenient to compute it from the formula

$$\text{(2.3.2)} \qquad \operatorname{grade}_R M = \inf\left\{ \operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M \right\}.$$

By the preceding expression we can find $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $\operatorname{grade}_R M = \operatorname{depth} R_{\mathfrak{p}}$, and deduce the first two inequalities below from the first two in (2.3.1), applied to $M = R$:

$$\text{(2.3.3)} \qquad \operatorname{depth} R - \dim_R M \leq \operatorname{grade}_R M \leq \operatorname{codim}_R M \leq \dim R - \dim_R M\,.$$

The third inequality above is obtained from the third one in (2.3.1), applied to $M = R$ and a prime $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $\dim_R M = \dim R/\mathfrak{p}$.

All three inequalities are, of course, well known. They show that the number

$$\gamma_R M = \dim R - \dim_R M - \operatorname{grade}_R M$$

is non-negative. Rewriting it in the form

$$\gamma_R M = \operatorname{cmd} R - (\operatorname{grade}_R M + \dim_R M - \operatorname{depth} R)$$

we conclude from (2.3.3) that

$$\text{(2.3.4)} \qquad 0 \leq \dim R - \dim_R M - \operatorname{grade}_R M \leq \operatorname{cmd} R\,.$$

Note that $\gamma_R M$ itself is the sum of two non-negative integers, namely:

$$\gamma_R M = (\dim R - \dim_R M - \operatorname{codim}_R M) + (\operatorname{codim}_R M - \operatorname{grade}_R M)\,.$$

Thus, (2.3.4) contains the fact that over a Cohen–Macaulay ring R the codimension and the grade of a module are both equal to its “naïve codimension” $\dim R - \dim M$.

(2.4) *Finite projective dimension.* Let $N \neq 0$ be a finite module of finite projective dimension over a local ring Q. The New Intersection Theorem of Peskine and Szpiro [28], Hochster [19], and P. Roberts [30], [32] yields an inequality

$$\dim Q \leq \dim_Q N + \operatorname{pd}_Q N\,, \tag{2.4.1}$$

cf. [31, pp. 69 and 74]. By the Auslander–Buchsbaum Equality it is equivalent to

$$\operatorname{cmd} Q \leq \operatorname{cmd}_Q N\,. \tag{2.4.2}$$

Now we can prove the second inequality of (2.1).

Proof of (2.1.b). Let $\mathfrak{p} \in \operatorname{Supp}_R M$ be such that $\operatorname{codim}_R M = \dim R_\mathfrak{p}$ and let $\mathfrak{q} \in \operatorname{Spec} Q$ be the inverse image of $\mathfrak{p}$. By using (2.4.1) we get:

$$\operatorname{codim}_Q M \leq \dim Q_\mathfrak{q} \leq \dim R_\mathfrak{p} + \operatorname{pd}_{Q_\mathfrak{q}} R_\mathfrak{q} \leq \operatorname{codim}_R M + \operatorname{pd}_Q R\,. \qquad \square$$

Next we use the New Intersection Theorem to show that the grade of a module of finite projective dimension is actually *equal* to its codimension.

(2.5) **Proposition.** *If M is a non-zero finite R–module of finite projective dimension, then* $\operatorname{grade}_R M = \operatorname{codim}_R M$*, and there exists a prime ideal* $\mathfrak{p}$ *minimal in* $\operatorname{Supp}_R M$ *such that the ring* $R_\mathfrak{p}$ *is Cohen–Macaulay of dimension equal to* $\operatorname{grade}_R M$.

Proof. By (2.3.2) we can choose $\mathfrak{q} \in \operatorname{Supp}_R M$ such that $\operatorname{grade}_R M = \operatorname{depth} R_\mathfrak{q}$. We then choose $\mathfrak{p} \in \operatorname{Spec} R$ contained in $\mathfrak{q}$ and minimal in $\operatorname{Supp}_R M$. By using the Auslander–Buchsbaum Equality we conclude from the choices of $\mathfrak{q}$ and $\mathfrak{p}$ that

$$\operatorname{grade}_R M = \operatorname{depth} R_\mathfrak{q} \geq \operatorname{pd}_{R_\mathfrak{q}} M_\mathfrak{q} \geq \operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} \geq \operatorname{grade}_R M,$$

hence $\operatorname{grade}_R M = \operatorname{depth} R_\mathfrak{p}$. Since $M_\mathfrak{p}$ is an $R_\mathfrak{p}$–module of finite length and of finite projective dimension, the ring $R_\mathfrak{p}$ is Cohen–Macaulay by (2.4.2).

Thus, $\operatorname{grade}_R M = \dim R_\mathfrak{p}$, and $\mathfrak{p}$ has been found. To finish the proof, observe that

$$\operatorname{grade}_R M = \dim R_\mathfrak{p} \geq \operatorname{codim}_R M \geq \operatorname{grade}_R M\,. \qquad \square$$

(2.6) *Imperfection.* Recall that the grade of M is equal to the smallest $i \in \mathbb{Z}$ such that $\operatorname{Ext}^i_R(M,R) \neq 0$, hence $\operatorname{grade}_R M \leq \operatorname{pd}_R M$. When equality holds the module M is said to be *perfect*. In general, we call the non-negative number

$$\operatorname{imp}_R M = \operatorname{pd}_R M - \operatorname{grade}_R M$$

the *imperfection* of M. The first inequality below follows from (2.4.1), and the second one is a consequence of the Auslander–Buchsbaum Equality, (2.4.1), and (2.3.4).

(2.6.1) If $\operatorname{pd}_R M$ is finite, then

$$\dim R - \dim_R M - \operatorname{grade}_R M \leq \operatorname{imp}_R M \leq \operatorname{cmd}_R M\,.$$

Perfect modules and Cohen–Macaulay modules have always been perceived as close kin. The precise relationship is described in the next statement. The implication (i) ⇒ (ii) is classical and follows immediately from the Auslander–Buchsbaum Equality. The converse is obtained by applying (2.4.2) and (2.6.1) (hence the New Intersection Theorem).

(2.6.2) The following conditions on M are equivalent:

(i) R is Cohen–Macaulay and M is perfect.
(ii) M is Cohen–Macaulay and $\operatorname{pd}_R M$ is finite.

The next two propositions will be proved concurrently. Their parts (c) contain the last preparatory results for the proof of (2.1.c). Parts (a) and (b) have been included for symmetry, and because partial results in these direction appear in the literature, cf. (2.9).

(2.7) **Proposition.** *If* $\operatorname{pd}_Q R$ *is finite, then there are inequalities:*

(a) $$\operatorname{grade}_R M + \operatorname{grade}_Q R \leq \operatorname{grade}_Q M\,.$$

(b) $$\operatorname{grade}_Q M \leq \operatorname{grade}_R M + \operatorname{pd}_Q R\,.$$

(c) $$\operatorname{grade}_Q M \leq \operatorname{pd}_R M + \operatorname{grade}_Q R\,.$$

(2.8) **Proposition.** *If* $\operatorname{pd}_Q R$ *and* $\operatorname{pd}_R M$ *are both finite, then there are inequalities:*

(a) $$\operatorname{imp}_Q M \leq \operatorname{imp}_R M + \operatorname{imp}_Q R\,.$$

(b) $$\operatorname{imp}_R M \leq \operatorname{imp}_Q M\,.$$

(c) $$\operatorname{imp}_Q R \leq \operatorname{imp}_Q M\,.$$

Proof of (2.7.a). Choose by (2.3.2) a prime $\mathfrak{q} \in \operatorname{Supp}_Q M$ such that $\operatorname{grade}_Q M = \operatorname{depth} Q_{\mathfrak{q}}$, and set $\mathfrak{p} = \mathfrak{q}R \in \operatorname{Supp}_R M$. As $\operatorname{depth} Q_{\mathfrak{q}} = \operatorname{depth} R_{\mathfrak{p}} + \operatorname{pd}_{Q_{\mathfrak{q}}} R_{\mathfrak{q}}$ by the Auslander–Buchsbaum Equality, we get:

$$\operatorname{grade}_Q M = \operatorname{depth} R_{\mathfrak{p}} + \operatorname{pd}_{Q_{\mathfrak{q}}} R_{\mathfrak{q}} \geq \operatorname{depth} R_{\mathfrak{p}} + \operatorname{grade}_{Q_{\mathfrak{q}}} R_{\mathfrak{q}} \geq \operatorname{grade}_R M + \operatorname{grade}_Q R\,. \qquad \square$$

Proof of (2.7.b). Choose a prime $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $\operatorname{grade}_R M = \operatorname{depth} R_{\mathfrak{p}}$, and let $\mathfrak{q} = \mathfrak{p} \cap Q$ be its inverse image in Q. The second inequality represents another application of the Auslander–Buchsbaum Equality:

$$\operatorname{grade}_Q M \leq \operatorname{depth} Q_{\mathfrak{q}} = \operatorname{depth} R_{\mathfrak{p}} + \operatorname{pd}_{Q_{\mathfrak{q}}} R_{\mathfrak{q}} \leq \operatorname{grade}_R M + \operatorname{pd}_Q R\,. \qquad \square$$

Proof of (2.8.a) *and* (2.8.b). It is well known that there is an equality

(2.8.1) $$\operatorname{pd}_Q M = \operatorname{pd}_R M + \operatorname{pd}_Q R\,,$$

and thus the desired formulas result from the corresponding parts of (2.7). □

Proof of (2.8.c). First note the canonical isomorphism

$$M \simeq M \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\mathrm{Hom}_R(R,R)\,.$$

As the projective dimension of of the R–module M is finite, so is that of the complex of R–modules $M^* = \mathbf{R}\mathrm{Hom}_R(M,R)$. It follows that the canonical morphisms

$$M \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\mathrm{Hom}_R(R,R) \to \mathbf{R}\mathrm{Hom}_R(M^*,R)$$
$$\mathbf{R}\mathrm{Hom}_Q\big(\mathbf{R}\mathrm{Hom}_R(M^*,R),Q\big) \leftarrow M^* \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\mathrm{Hom}_Q(R,Q)$$

are isomorphisms in the derived category, cf. e.g. [2, (4.4.I)]. Thus we obtain equalities

$$\operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(M,Q) = \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q\big(\mathbf{R}\mathrm{Hom}_R(M^*,R),Q\big) = \operatorname{amp}\big(M^* \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\mathrm{Hom}_Q(R,Q)\big)\,.$$

Next we apply Iversen's Amplitude Inequality [21, (3.2)]: although the quoted text requires that R be an algebra over some field, the proof uses this assumption only through a reference to the New Intersection Theorem, hence due to [32] the result is now available without restrictions on R. It yields an inequality

$$\operatorname{amp}\big(M^* \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\mathrm{Hom}_Q(R,Q)\big) \geq \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(R,Q)\,,$$

so that we get

$$\operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(M,Q) \geq \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(R,Q)\,.$$

Finally, as $\operatorname{pd}_Q M$ and $\operatorname{pd}_Q R$ are both finite, we have

$$\operatorname{imp}_Q M = \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(M,Q)\ ;\ \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(R,Q) = \operatorname{imp}_Q R\,.$$ □

Proof of (2.7.c). The inequality $\operatorname{grade}_Q M \leq \operatorname{pd}_R M + \operatorname{grade}_Q R$ follows directly from (2.8.c) and (2.8.1). □

We are finally ready to establish the last inequality in (2.1).

Proof of (2.1.c). This is a consequence of (2.7.c), because by (2.5) grade and height coincide for annihilators of finite modules of finite projective dimension. □

(2.9) *Remark.* The inequality in (2.7.b) above is obtained in [9, (16.17)] under the additional assumption that the R–module M is a perfect. When this condition holds, (2.7.a) and (2.7.b) show that actually there is an equality

$$\operatorname{grade}_Q M = \operatorname{grade}_R M + \operatorname{grade}_Q R\,,$$

which in view of (2.5) can be rewritten as an equality of codimensions

$$\operatorname{codim}_Q M = \operatorname{codim}_R M + \operatorname{codim}_Q R\,.$$

3. Cohen–Macaulay defect of a local homomorphism

The term *local homomorphism* denotes a homomorphism of *local* rings which maps the maximal ideal of the source into that of the target. For the rest of the section we fix a local homomorphism

$$\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},\ell)\,.$$

The local ring $S/\mathfrak{m}S$ is called the *closed fiber* of φ. The flat dimension $\operatorname{fd}_R S$ of the R–module S is also denoted $\operatorname{fd}\varphi$; we refer to this number as to the *flat dimension* of φ.

Certain types of local homomorphisms are designated by special names.

(3.1.1) *Structure homomorphism.* The unique homomorphism $\eta_S\colon \mathbb{Z}_{(p)} \to S$, where $p = \operatorname{char}\ell$, is called the *local structure homomorphism* of the local ring S.

(3.1.2) *Completion.* We denote by $\rho\colon R \to \widehat{R}$ and $\sigma\colon S \to \widehat{S}$ the canonical maps of R and S to their $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completion, respectively. The homomorphism φ induces by continuity a map $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$, called the *completion* of φ; it is a local homomorphism. We also consider the *semi-completion* of φ, that is, the local homomorphism $\grave{\varphi} = \widehat{\varphi}\rho = \sigma\varphi\colon R \to \widehat{S}$. As noted in [6, (3.3)], there are equalities

$$\operatorname{fd}\varphi = \operatorname{fd}\grave{\varphi} = \operatorname{fd}\widehat{\varphi}\,.$$

(3.1.3) *Cohen factorization.* It is proved in [6, (1.1)] that $\grave{\varphi}$ embeds into a commutative triangle of local homomorphisms of local rings

$$\begin{array}{ccccc} & & R' & & \\ & \dot{\varphi}\nearrow & & \searrow\varphi' & \\ R & & \xrightarrow[\grave{\varphi}]{\qquad} & & \widehat{S} \end{array}$$

such that $\dot{\varphi}$ is flat, R' is complete, $R'/\mathfrak{m}R'$ is regular, and φ' is surjective. Any such diagram is called a *Cohen factorization* of $\grave{\varphi}$. By [5, (2.7)], cf. also [6, (3.3)] there are inequalities

$$\operatorname{fd}\varphi \le \operatorname{fd}\varphi' \le \operatorname{fd}\varphi + \dim R' - \dim R\,,$$

which in particular show that $\operatorname{fd}\varphi$ and $\operatorname{fd}\varphi'$ are finite simultaneously.

Some numerical invariants defined in [6] play a crucial role in this paper.

(3.2) *Numerical invariants of local homomorphisms.* It is shown in [6, (2.1)] that the number

$$\dim\varphi = \dim R' - \dim R - \operatorname{codim}_{R'}\widehat{S}$$

is an invariant of φ, called its *dimension*. Unlike the corresponding notion for rings, it may take on negative values, as a reflection of the structure of φ.

The *depth* of φ is introduced in [6, (2.2)] by

$$\operatorname{depth}\varphi = \operatorname{depth}S - \operatorname{depth}R\,.$$

In case $\operatorname{fd}\varphi$ is finite, $\operatorname{depth}\varphi$ is defined in [8] in a completely different way, but [*idem.*, (5.3)] shows that that definition agrees with the present one.

The *Cohen–Macaulay defect* of φ, cf. [6, (2.3)], is defined to be the difference

$$\operatorname{cmd}\varphi = \dim\varphi - \operatorname{depth}\varphi\,.$$

It may be negative; for instance, the canonical projection $\varepsilon\colon R \to k$ has $\operatorname{cmd}\varepsilon = -\operatorname{cmd} R$.

We need some properties of these invariants.

(3.3) *Remark.* It is easy to see from the definitions that the dimension, depth, and Cohen–Macaulay defect of φ coincide with the corresponding invariants of $\grave{\varphi}$ and of $\widehat{\varphi}$.

(3.4) *Remark.* If the ring R is Cohen–Macaulay, then $\dim\varphi = \dim S - \dim R$ and $\operatorname{cmd}\varphi = \operatorname{cmd} S$; in particular, $\operatorname{cmd}\varphi \geq 0$, and the local structure homomorphism has $\operatorname{cmd}\eta_S = \operatorname{cmd} S$.

Indeed, then $\grave{\varphi}$ is a flat local homomorphism with Cohen–Macaulay base and closed fiber, hence the local ring R' is Cohen–Macaulay; thus, $\dim S = \dim \widehat{S} = \dim R' - \operatorname{codim}_{R'} \widehat{S}$, so that $\dim\varphi = \dim S - \dim R$.

(3.5) **Proposition.** *If φ is flat, then*

$$\operatorname{cmd}\varphi = \operatorname{cmd}(S/\mathfrak{m}S) = \operatorname{cmd} S - \operatorname{cmd} R\,.$$

Proof. The equalities are obtained in [6, (2.8)] from a more general result. A direct proof may be found in [4, (2.4)]. A third proof is given at the end of this section. □

Next we focus on homomorphisms of finite flat dimension.

(3.6) **Lemma.** *When φ is surjective* $\dim\varphi = -\operatorname{codim}_R S$. *If furthermore* $\operatorname{fd}\varphi < \infty$, *then*

$$\begin{aligned}\dim\varphi &= -\operatorname{grade}_R S\,;\\ \operatorname{depth}\varphi &= -\operatorname{pd}_R S = -\operatorname{fd}\varphi\,;\\ \operatorname{cmd}\varphi &= \operatorname{imp}_R S\,.\end{aligned}$$

Proof. First note that $R \xrightarrow{\rho} \widehat{R} \xrightarrow{\widehat{\varphi}} \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$, hence $\dim\varphi = -\operatorname{codim}_{\widehat{R}} \widehat{S} = -\operatorname{codim}_R S$. When $\operatorname{fd}\varphi$ is finite the expression for $\dim\varphi$ follows from here and (2.5), the one for $\operatorname{depth}\varphi$ is a consequence of the Auslander–Buchsbaum Equality, and the formula for $\operatorname{cmd}\varphi$ results from the other two.

(3.7) **Proposition.** *If $R \to R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$, then* $\operatorname{cmd}\varphi = \operatorname{cmd}\varphi'$. *If furthermore* $\operatorname{fd}\varphi < \infty$, *then also*

$$\operatorname{depth}\varphi = \dim R' - \dim R - \operatorname{pd}_{R'} \widehat{S}\,;$$

$$\operatorname{cmd}\varphi = \operatorname{imp}_{R'} \widehat{S}\,;$$

in particular, $\operatorname{imp}_{R'} \widehat{S}$ *does not depend on the choice of the factorization, and* $\operatorname{cmd}\varphi \geq 0$.

Proof. The definition of depth shows that $\operatorname{depth}\varphi = \operatorname{depth} R' - \operatorname{depth} R + \operatorname{depth}\varphi'$. Furthermore, $\operatorname{depth} R' - \operatorname{depth} R = \dim R' - \dim R$ because $\dot{\varphi}$ is flat with regular fiber. From the definition of Cohen–Macaulay defect and the lemma we now conclude that $\operatorname{cmd}\varphi = \operatorname{codim}_{R'} \widehat{S} = \operatorname{cmd}\varphi'$.

When $\operatorname{fd}\varphi$ is finite, so is $\operatorname{fd}\varphi'$ by (3.1.3), hence the lemma yields $\operatorname{depth}\varphi' = -\operatorname{pd}_{R'} \widehat{S}$. The expression for $\operatorname{depth}\varphi$ and the second formula for $\operatorname{cmd}\varphi$ follow. □

In some cases the Cohen–Macaulay defect of a homomorphism is equal to the Cohen–Macaulay defect (1.1) of the complex obtained by base change of a dualizing complex.

(3.8) **Proposition.** *Let* $\tau\colon Q \to T$ *be a local homomorphism. If* $\operatorname{fd}\tau$ *is finite and* C *is a dualizing complex for* Q*, then*

$$\operatorname{cmd}\tau = \operatorname{cmd}_T(C \overset{\mathbf{L}}{\otimes}_Q T)\,.$$

Proof. First we reduce to the case of a complete local ring T. Indeed the equalities

$$\operatorname{cmd}_T(C \overset{\mathbf{L}}{\otimes}_Q T) = \operatorname{cmd}_{\widehat{T}}\big((C \overset{\mathbf{L}}{\otimes}_Q T) \otimes_T \widehat{T}\big) = \operatorname{cmd}_{\widehat{T}}(C \overset{\mathbf{L}}{\otimes}_Q \widehat{T})\,,$$

show the right hand side of the formula to be proved does not change if we replace T by $\widehat{T}$. By (3.1.2) and (3.3) the other side has the same property with respect to τ and $\grave{\tau}$.

Next we reduce to the case of a surjective homomorphism τ. Taking a Cohen factorization $Q \to Q' \xrightarrow{\tau'} T$ of τ we note that $\operatorname{fd}\tau'$ is finite, cf. (3.1.3). The complex of Q'–modules $C \otimes_Q Q'$ is dualizing for Q', cf. (1.3.4). The canonical isomorphism $C \overset{\mathbf{L}}{\otimes}_Q T \simeq (C \otimes_Q Q') \overset{\mathbf{L}}{\otimes}_{Q'} T$ shows that we can replace Q by Q' on the right hand side of the desired equality. By (3.7) we can simultaneously replace τ by τ' on the left.

From now on we assume that τ is surjective. By (3.6) we have to prove that

$$\operatorname{imp}_Q T = \operatorname{cmd}_T(C \overset{\mathbf{L}}{\otimes}_Q T)\,.$$

As $\operatorname{pd}_Q T$ is finite we have $\operatorname{imp}_Q T = \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(T, Q)$. The canonical isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_Q(T,Q) &\simeq \mathbf{R}\mathrm{Hom}_Q\big(T, \mathbf{R}\mathrm{Hom}_Q(C,C)\big) \simeq \mathbf{R}\mathrm{Hom}_Q(T \overset{\mathbf{L}}{\otimes}_Q C, C)\\ &\simeq \mathbf{R}\mathrm{Hom}_Q\big(C, \mathbf{R}\mathrm{Hom}_Q(T,C)\big) = \mathbf{R}\mathrm{Hom}_Q(C,D)\,,\end{aligned}$$

where D stands for the complex $\mathbf{R}\mathrm{Hom}_Q(T,C)$, imply that

$$\operatorname{imp}_Q T = \operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(C,D)\,.$$

To obtain the desired equality note that D is a dualizing complex for T, cf. (1.3.3), and then apply (1.3.2) to get

$$\operatorname{amp} \mathbf{R}\mathrm{Hom}_Q(C, D) = \operatorname{cmd}_T(C \overset{\mathbf{L}}{\otimes}_Q T)\,. \qquad \square$$

As a first application of the proposition we give the announced

Proof of (3.5). The closed fiber of $\widehat{\varphi}$ is canonically isomorphic to the completion of that of φ, and hence has the same Cohen–Macaulay defect; thus, by (3.1.2) and (3.3) we can assume R and S are complete. In particular, R has a dualizing complex D, cf. (1.3.1).

As S is a flat R–module, the functors $- \overset{\mathbf{L}}{\otimes}_R S$ and $- \otimes_R S$ are naturally equivalent, so (3.8), (1.2), and (1.3.2) yield

$$\operatorname{cmd} \varphi = \operatorname{cmd}_S(D \otimes_R S) = \operatorname{cmd}_R D + \operatorname{cmd}(S/\mathfrak{m}S) = \operatorname{amp} \mathbf{R}\mathrm{Hom}_R(D, D) + \operatorname{cmd}(S/\mathfrak{m}S)\,.$$

The isomorphism $\mathbf{R}\mathrm{Hom}_R(D, D) \simeq R$ from the definition of dualizing complex shows that $\operatorname{amp} \mathbf{R}\mathrm{Hom}_R(D, D) = 0$, and we get $\operatorname{cmd} \varphi = \operatorname{cmd}(S/\mathfrak{m}S)$, as desired. $\square$

4. Composition of local homomorphisms

The existence of strong relations between the numerical invariants of local homomorphisms of finite flat dimension and those of their composition opens up a new perspective for the study of both rings and maps. It engulfs the powerful techniques available earlier only for flat homomorphisms and extends them to a much wider class of maps.

(4.1) **Theorem.** *If $\psi\colon Q \to R$ and $\varphi\colon R \to S$ are local homomorphisms, then*

$$\dim \varphi\psi \le \dim \psi + \dim \varphi\,;$$
$$\operatorname{cmd} \varphi\psi \le \operatorname{cmd} \psi + \operatorname{cmd} \varphi\,,$$

and equalities hold when $\operatorname{fd} \psi$ *is finite and φ is flat.*

We know of no example of homomorphisms φ and ψ, *both of finite flat dimension*, for which a strict inequality appears in the theorem. The next result lends further support to the conjecture that for such homomorphisms there always is equality. Such a conjecture is equivalent to the statement that an equality $\dim R/\mathfrak{a} + \operatorname{ht} \mathfrak{a} = \dim R$ holds whenever $\mathfrak{a}$ is an ideal of finite projective dimension in a local ring R: this is a special case of a well known conjecture of M. Auslander, discussed in more detail in the Appendix.

(4.2) **Theorem.** *If $\psi\colon Q \to R$ is a local homomorphism of finite flat dimension, then*

$$\operatorname{depth} \psi + \dim \varphi \le \dim \varphi\psi\,;$$
$$\operatorname{cmd} \varphi \le \operatorname{cmd} \varphi\psi\,,$$

for any local homomorphism $\varphi\colon R \to S$. If also the flat dimension of φ is finite, then

$$\dim \psi + \operatorname{depth} \varphi \le \dim \varphi\psi\,;$$
$$\operatorname{cmd} \psi \le \operatorname{cmd} \varphi\psi\,.$$

In view of (3.4), for $\psi = \eta_R$ the theorems reduce to earlier results, cf. [6, (2.4) and (3.3)].

(4.3) **Corollary.** *If* $\varphi\colon R \to S$ *is a local homomorphism, then*

$$\begin{gathered}\operatorname{depth} R + \dim \varphi \leq \dim S \leq \dim R + \dim \varphi\,;\\ \operatorname{cmd} \varphi \leq \operatorname{cmd} S \leq \operatorname{cmd} R + \operatorname{cmd} \varphi\,.\end{gathered}$$

If furthermore $\operatorname{fd} \varphi$ *is finite, then the following also hold*

$$\begin{gathered}\dim R + \operatorname{depth} \varphi \leq \dim S\,;\\ \operatorname{cmd} R \leq \operatorname{cmd} S\,.\end{gathered} \qquad \square$$

The proofs take up the rest of this section. Two reductions are available immediately. By (3.1.2) and (3.3) we can assume that all rings are complete. As by definition (3.2) depth is additive on compositions of homomorphisms in the sense that

$$\operatorname{depth} \varphi\psi = \operatorname{depth} \psi + \operatorname{depth} \varphi\,,$$

it suffices to prove one of the (in)equalities from each pair. Most of the arguments rely on a specific construction of Cohen factorizations of ψ, φ, and $\varphi\psi$.

(4.4) **Construction.** Let $\psi\colon Q \to R$ and $\varphi\colon R \to S$ be local homomorphisms of local rings, with R and S complete. Assume that

(4.4.1) $Q \xrightarrow{\dot\psi} Q' \xrightarrow{\psi'} R$ is a Cohen factorization of ψ,

where Q' is a complete local ring with maximal ideal $\mathfrak{l}'$.

There is then a commutative diagram of local homomorphisms of local rings

$$\begin{array}{ccccccccc} & & & & Q'' & & & & \\ & & & {\scriptstyle\dot\zeta}\nearrow & & \searrow{\scriptstyle\zeta'} & & & \\ & & Q' & & & & R' & & \\ & {\scriptstyle\dot\psi}\nearrow & & \searrow{\scriptstyle\psi'} & & {\scriptstyle\dot\varphi}\nearrow & & \searrow{\scriptstyle\varphi'} & \\ Q & & \xrightarrow[\psi]{\quad\quad} & & R & & \xrightarrow[\varphi]{\quad\quad} & & S \end{array}$$

with the following properties:

(4.4.2) $\varphi'\dot\varphi$ is a Cohen factorization of φ.
(4.4.3) $(\varphi'\zeta')(\dot\zeta\dot\psi)$ is a Cohen factorization of $\varphi\psi$.
(4.4.4) $R' = R \otimes_{Q'} Q''$, $\dot\varphi = R \otimes_{Q'} \dot\zeta$, and $\zeta' = \psi' \otimes_{Q'} Q''$.
(4.4.5) $\dot\zeta$ is flat, its closed fiber $Q''/\mathfrak{l}'Q''$ is regular and isomorphic to that of $\dot\varphi'$.

As a first step, choose a Cohen factorization $Q' \xrightarrow{\dot\zeta} Q'' \to S$ of $\varphi\psi'$. By setting $R' = R \otimes_{Q'} Q''$, $\dot\varphi = R \otimes_{Q'} \dot\zeta$, and $\zeta' = \psi' \otimes_{Q'} Q''$, we meet the conditions of (4.4.4).

Finally, let $\varphi'\colon R\otimes_{Q'} Q'' \to S$ be the canonical homomorphism defined by $\varphi\psi'$ and $\dot{\zeta}$. The resulting diagram is clearly commutative.

The homomorphism $\varphi'\zeta'$ is surjective by construction. The homomorphism $\dot{\zeta}\dot{\psi}$ is flat as a composition of flat homomorphisms. For (4.4.3) it remains to show the regularity of $k\otimes_Q Q''$. This can be seen by noting that the base of the flat local homomorphism $k\otimes_Q \dot{\zeta}\colon k\otimes_Q Q' \to k\otimes_Q Q''$ is regular by construction, and its closed fiber is regular since it is canonically isomorphic to $k\otimes_{Q'} Q''$.

To establish (4.4.5) note that by construction the homomorphism $\dot{\zeta}$ is flat with regular fiber, and that the sequence of isomorphisms

$$Q''/\mathfrak{l}'Q'' \cong k\otimes_{Q'} Q'' \cong k\otimes_R (R\otimes_{Q'} Q'') \cong k\otimes_R R' \cong R'/\mathfrak{m}R'$$

identifies this fiber with that of $\dot{\varphi}$. To finish with (4.4.2) remark that the surjectivity of φ' follows from that of $\varphi'\zeta'$ and the flatness of $\dot{\varphi}$ is a consequence of that of $\dot{\zeta}$.

Next we assemble some numerical data pertaining to the preceding construction:

$$\dim Q'' - \dim Q' = \dim R' - \dim R\,. \tag{4.5.1}$$

$$\operatorname{codim}_{Q''} R' = \operatorname{codim}_{Q'} R\,. \tag{4.5.2}$$

$$\operatorname{pd}_{Q''} R' = \operatorname{pd}_{Q'} R\,. \tag{4.5.3}$$

Indeed, the first equality is immediate from (4.4.5). To see the second one, note that the isomorphism $R' \cong Q''/(\operatorname{Ker}\psi')Q''$ shows $\operatorname{codim}_{Q''} R'$ is equal to the height of $(\operatorname{Ker}\psi')Q''$, which is the same as that of $\operatorname{Ker}\psi'$, cf. [22, (13.B.3)]. For the third one it suffices to remark that by tensoring with Q'' over Q' a minimal free resolution of the Q'–module R one gets a minimal free resolution of the Q''–module $R' = R\otimes_{Q'} Q''$.

We are now ready for the proofs of the theorems.

Proof of (4.1). The inequality of dimensions results from the computation

$$\begin{aligned}
\dim\varphi\psi &= \dim Q'' - \dim Q - \operatorname{codim}_{Q''} S && [(4.4.3)]\\
&\le \dim Q'' - \dim Q - \operatorname{codim}_{Q''} R' - \operatorname{codim}_{R'} S && [(2.1.\text{a})]\\
&= \dim Q'' - \dim Q - \operatorname{codim}_{Q'} R - \operatorname{codim}_{R'} S && [(4.5.2)]\\
&= (\dim Q' - \dim Q - \operatorname{codim}_{Q'} R) + (\dim Q'' - \dim Q' - \operatorname{codim}_{R'} S)\\
&= \dim\psi + (\dim Q'' - \dim Q' - \operatorname{codim}_{R'} S) && [(4.4.1)]\\
&= \dim\psi + (\dim R' - \dim R - \operatorname{codim}_{R'} S) && [(4.5.1)]\\
&= \dim\psi + \dim\varphi && [(4.4.2)].
\end{aligned}$$

In order to establish the equality of Cohen–Macaulay defects when ψ has finite flat dimension and φ is flat, consider a dualizing complex C for the complete local ring Q, cf.

(1.3.1). The desired equality is obtained between both ends of the following sequence:

$$\begin{aligned}
\operatorname{cmd}\varphi\psi &= \operatorname{cmd}_S(C \overset{\mathbf{L}}{\otimes}_Q S) && [(3.8)]\\
&= \operatorname{cmd}_S\big((C \overset{\mathbf{L}}{\otimes}_Q R) \otimes_R S\big)\\
&= \operatorname{cmd}_R(C \overset{\mathbf{L}}{\otimes}_Q R) + \operatorname{cmd}(S/\mathfrak{m}S) && [(1.2)]\\
&= \operatorname{cmd}\psi + \operatorname{cmd}(S/\mathfrak{m}S) && [(3.8)]\\
&= \operatorname{cmd}\psi + \operatorname{cmd}\varphi && [(3.5)]. \quad \square
\end{aligned}$$

Proof of (4.2). By hypothesis, $\operatorname{fd}\psi$ is finite

For the first assertion, we prove the inequality involving dimensions and depth, starting with the sequence of (in)equalities:

$$\begin{aligned}
\dim\varphi\psi &= \dim Q'' - \dim Q - \operatorname{codim}_{Q''} S && [(4.4.3)]\\
&\geq \dim Q'' - \dim Q - \operatorname{pd}_{Q''} R' - \operatorname{codim}_{R'} S && [(2.1.b)]\\
&= \dim Q'' - \dim Q - \operatorname{pd}_{Q'} R - \operatorname{codim}_{R'} S && [(4.5.3)]\\
&= (\dim Q' - \dim Q - \operatorname{pd}_{Q'} R) + (\dim Q'' - \dim Q' - \operatorname{codim}_{R'} S)\\
&= (\dim Q' - \dim Q - \operatorname{pd}_{Q'} R) + (\dim R' - \dim R - \operatorname{codim}_{R'} S) && [(4.5.1)]\\
&= (\dim Q' - \dim Q - \operatorname{pd}_{Q'} R) + \dim\varphi && [(4.4.2)].
\end{aligned}$$

It remains to note that $\dim Q' - \dim Q - \operatorname{pd}_{Q'} R = \operatorname{depth}\psi$, due to (3.1.3) and (4.4.1).

To establish the inequality of Cohen–Macaulay defects in the second assertion, remark that $\operatorname{fd}\zeta' = \operatorname{fd}\psi'$ by (4.5.3). It thus follows from (3.1.3) that $\operatorname{fd}\zeta'$ and $\operatorname{fd}\varphi'$ are both finite. By (3.6) the desired inequality amounts to $\operatorname{imp}_{Q''} S \geq \operatorname{imp}_{R'} S$, which is (2.8.c). $\square$

5. Localization and formal fibers

For a noetherian ring the Cohen–Macaulay condition may be introduced either by requiring the localizations at all *maximal* ideals to be Cohen–Macaulay local rings, or by requiring this property to hold at all *prime* ideals. An argument is then needed to see that both conditions produce the same concept.

More generally, for prime ideals $\mathfrak{q} \subseteq \mathfrak{n}$ in an arbitrary noetherian ring S there is an inequality $\operatorname{cmd} S_{\mathfrak{q}} \leq \operatorname{cmd} S_{\mathfrak{n}}$, cf. [16, (6.11.5)]. Thus, the supremum of $\operatorname{cmd} S_{\mathfrak{q}}$ when $\mathfrak{q}$ ranges over the $\operatorname{Spec} S$ coincides with the one taken over the maximal spectrum $\operatorname{Max} S$. For a local ring the number so obtained coincides with the one defined earlier; for a general noetherian ring S it provides a measure of the deviation of S from being Cohen–Macaulay. We call this number the *Cohen–Macaulay defect* of S, and denote it $\operatorname{cmd} S$.

Following the general philosophy of this paper, we extend the notion to homomorphisms of noetherian rings. To do this we first need a notion of localization.

(5.1) *Localization.* Let $\varphi : R \to S$ be a homomorphism of commutative rings. When $\mathfrak{q}$ is a prime ideal in S and $\mathfrak{q} \cap R$ is its inverse image in R, the induced local homomorphism

$$\varphi_{\mathfrak{q}} \colon R_{\mathfrak{q}\cap R} \longrightarrow S_{\mathfrak{q}}$$

is called the *localization* of φ at $\mathfrak{q}$. There always is an inequality

$$\operatorname{fd} \varphi_{\mathfrak{q}} \leq \operatorname{fd} \varphi\,. \tag{5.1.1}$$

For $\mathfrak{p} \in \operatorname{Spec} R$ we set as usual $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, and consider the *fiber* $k(\mathfrak{p}) \otimes_R S$ of φ at $\mathfrak{p}$. When $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in \operatorname{Spec} S$ we identify the localization $\big(k(\mathfrak{p}) \otimes_R S\big)_{\widetilde{\mathfrak{q}}}$ of the fiber at the image $\widetilde{\mathfrak{q}}$ of $\mathfrak{q}$ with the closed fiber $k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ of the local homomorphism $\varphi_{\mathfrak{q}}$.

At this point a notable distinction appears in the behavior of Cohen–Macaulay defects for rings and for homomorphisms: the desirable inequality $\operatorname{cmd} \varphi_{\mathfrak{q}} \leq \operatorname{cmd} \varphi_{\mathfrak{n}}$ for prime ideals $\mathfrak{q}$ and $\mathfrak{n}$ in S with $\mathfrak{q} \subseteq \mathfrak{n}$ may fail even for a faithfully flat homomorphisms $\varphi \colon R \to S$. To see this, consider Ogoma's [25, Theorem 5] construction of a local domain R for which $\widehat{R}$ has a prime ideal $\mathfrak{q}$ lying over (0) in R, with $\widehat{R}_{\mathfrak{q}}$ not equidimensional and thus not Cohen–Macaulay. It follows from (3.5) that the completion homomorphism $\rho \colon R \to \widehat{R}$ has $\operatorname{cmd} \rho_{\mathfrak{q}} = \operatorname{cmd} \widehat{R}_{\mathfrak{q}} > 0 = \operatorname{cmd} \rho$.

It is proved in [6, (3.6)] that the Cohen–Macaulay defect of a homomorphism essentially of finite type localizes properly. We sketch the argument both for completeness, and because it is needed later on.

(5.2) **Proposition.** *If $\varphi \colon R \to S$ is a local homomorphism essentially of finite type and of finite flat dimension, then for each $\mathfrak{q} \in \operatorname{Spec} S$ the homomorphism $\varphi_{\mathfrak{q}}$ has the same properties, and* $\operatorname{cmd} \varphi_{\mathfrak{q}} \leq \operatorname{cmd} \varphi$. □

Proof. As φ is essentially of finite type, it factors as $\zeta\xi$, with ξ flat local with regular fibers and ζ surjective. This induces a factorization $\varphi_{\mathfrak{q}} = \zeta_{\mathfrak{p}'}\xi_{\mathfrak{q}}$ with similar properties. Furthermore, the equalities $\grave{\varphi} = \widehat{\zeta}\xi$ and $\grave{}(\varphi_{\mathfrak{q}}) = (\zeta_{\mathfrak{p}'})\widehat{}\,\xi_{\mathfrak{q}}$ provide Cohen factorizations of $\grave{\varphi}$ and $\grave{}(\varphi_{\mathfrak{q}})$, respectively. By applying (3.7) and (3.3), we get

$$\operatorname{cmd} \varphi' = \operatorname{cmd} \widehat{\zeta} = \operatorname{cmd} \zeta\,;$$
$$\operatorname{cmd} \varphi_{\mathfrak{q}} = \operatorname{cmd} (\zeta_{\mathfrak{q}})\widehat{} = \operatorname{cmd} \zeta_{\mathfrak{q}}\,.$$

Thus, it suffices to treat the case when φ is surjective. By (3.6) we then have to show that for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} \supset \operatorname{Ker} \varphi$ there is an inequality $\operatorname{imp}_{R_{\mathfrak{p}}} S_{\mathfrak{p}} \leq \operatorname{imp}_R S$, which is elementary. □

Our main result on localization imposes no condition on the homomorphism. Among other things it implies that the only obstruction to proper localization is rooted in pathological behavior of the formal fibers of the source ring.

(5.3) **Theorem.** *Let $\varphi\colon R \to S$ be a local homomorphism of finite flat dimension. If $\mathfrak{q}$ is a prime ideal in S and $\mathfrak{p} = \mathfrak{q} \cap R$, then there is an inequality*

$$\operatorname{cmd} \varphi_{\mathfrak{q}} + \operatorname{cmd}\bigl(k(\mathfrak{q}) \otimes_S \widehat{S}\bigr) \le \operatorname{cmd} \varphi + \operatorname{cmd}\bigl(k(\mathfrak{p}) \otimes_R \widehat{R}\bigr)\,.$$

(5.4) *Remark.* Applying the theorem to the local structure homomorphism η_S, for each prime ideal $\mathfrak{q}$ of S we get in view of (3.4) an inequality

$$\operatorname{cmd} S_{\mathfrak{q}} + \operatorname{cmd}\bigl(k(\mathfrak{q}) \otimes_S \widehat{S}\bigr) \le \operatorname{cmd} S\,,$$

which can also be obtained by using the classical additivity of Cohen–Macaulay defects of rings on flat homomorphisms.

For the theorem we need the following known result. We include a self-contained version of the proof given in [4, §3, Step 1].

(5.5) **Lemma.** *Let $\varkappa\colon \widehat{R} \to R'$ be a flat local homomorphism of complete local rings, let $\mathfrak{p}'$ be a prime ideal in R', and set $\mathfrak{p}^* = \mathfrak{p}' \cap \widehat{R}$. If the closed fiber of $\varkappa$ is regular, then the local ring $\bigl(k(\mathfrak{p}^*) \otimes_{\widehat{R}} R'\bigr)_{\mathfrak{p}'}$ is a complete intersection.*

Proof. Choose by Cohen's Structure Theorem a surjection $\pi\colon Q \to \widehat{R}$ from a regular local ring Q with residue field k. In a Cohen factorization $Q \xrightarrow{\dot{\xi}} Q' \xrightarrow{\xi'} R'$ of $\xi = \varkappa\pi$ the complete local ring Q' is regular, and the induced map $k \otimes_Q \xi'\colon k \otimes_Q Q' \to k \otimes_Q R' \cong k \otimes_{\widehat{R}} R'$ is a surjection of regular local rings. Its kernel is therefore generated by a part $\bar{\mathbf{x}}$ of a regular system of parameters of $k \otimes_Q Q'$. Due to the equality $\operatorname{Ker}\bigl(k \otimes_Q \xi'\bigr) = (\operatorname{Ker} \xi')\bigl(k \otimes_Q Q'\bigr)$, we can lift $\bar{\mathbf{x}}$ to a part $\mathbf{x}$ of a regular system of parameters of Q', such that $\mathbf{x} \subset \operatorname{Ker} \xi'$.

With the induced maps $\ddot{\xi}\colon Q \to Q' \to Q'' = Q'/(\mathbf{x})$ and $\xi''\colon Q'' \to R'$ we now have a commutative square of local homomorphisms

$$\begin{array}{ccc} Q & \xrightarrow{\ \ddot{\xi}\ } & Q'' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \xi''} \\ \widehat{R} & \xrightarrow[\ \varkappa\ ]{} & R' \end{array}$$

in which the rings Q and Q'' are regular. The vertical maps are obviously surjective. The horizontal ones are flat (for $\ddot{\xi}$ this follows from [23, (22.5.Corollary)]), and their closed fibers are regular local rings which are canonically isomorphic by $k \otimes_Q \xi''$.

By base change we get from here a factorization $\varkappa = \dot{\varkappa}\varkappa'$, with a flat homomorphism $\dot{\varkappa} = \widehat{R} \otimes_Q \ddot{\xi}\colon \widehat{R} \to \widehat{R} \otimes_Q Q''$ and a surjective homomorphism $\varkappa'\colon \widehat{R} \otimes_Q Q'' \to R'$. In view of the flatness of the $\widehat{R}$–module R', the last map induces an exact sequence

$$0 \to k \otimes_{\widehat{R}} \operatorname{Ker} \varkappa' \to k \otimes_{\widehat{R}} \widehat{R} \otimes_Q Q'' \xrightarrow{k \otimes_{\widehat{R}} \varkappa'} k \otimes_{\widehat{R}} R' \to 0\,.$$

Since $k \otimes_{\widehat{R}} \varkappa' = k \otimes_Q \xi''$ is an isomorphism, it follows that $k \otimes_{\widehat{R}} \operatorname{Ker} \varkappa' = 0$, hence $\operatorname{Ker} \varkappa' = 0$ by Nakayama, and thus $\varkappa'$ is an isomorphism.

Set $\mathfrak{q}'' = \mathfrak{p}' \cap Q''$ and $\mathfrak{q} = \mathfrak{q}'' \cap Q$. The ring $\big(k(\mathfrak{p}^*) \otimes_{\widehat{R}} R'\big)_{\mathfrak{p}'}$ is canonically isomorphic with the closed fiber of $\ddot{\xi}_{\mathfrak{q}''} \colon Q_{\mathfrak{q}} \to Q''_{\mathfrak{q}''}$, which is the quotient of the regular ring $Q''_{\mathfrak{q}''}$ by the extension of the maximal ideal of the regular ring $Q_{\mathfrak{q}}$. This ideal is generated by a $Q_{\mathfrak{q}}$–regular sequence, which is also $Q''_{\mathfrak{q}''}$–regular due to the flatness of $\ddot{\xi}_{\mathfrak{q}''}$. □

Proof of (5.3). Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization of $\dot{\varphi}$. As the local ring R' is complete, $\dot{\varphi}$ factors through the completion of R, and we get a commutative diagram

$$
\begin{array}{ccccc}
R & & \xrightarrow{\quad\varphi\quad} & & S \\
\rho\big\downarrow & & & & \big\downarrow\sigma \\
\widehat{R} & \underset{\varkappa}{\longrightarrow} & R' & \underset{\varphi'}{\longrightarrow} & \widehat{S}
\end{array}
$$

of local homomorphisms of local rings, in which the vertical arrows are completion maps.

Take any $\widetilde{\mathfrak{q}} \in \operatorname{Spec}\big(k(\mathfrak{p}) \otimes_S \widehat{S}\big)$, and set $\mathfrak{q}^* = \widetilde{\mathfrak{q}} \cap \widehat{S}$, $\mathfrak{p}' = \mathfrak{q}^* \cap R'$, and $\mathfrak{p}^* = \mathfrak{p}' \cap \widehat{R}$. The diagram produces an equality $\sigma_{\mathfrak{q}^*}\varphi_{\mathfrak{q}} = \varphi'_{\mathfrak{q}^*}\varkappa_{\mathfrak{p}'}\rho_{\mathfrak{p}^*}$ of local homomorphisms. Remarking that $\operatorname{fd}\varphi' < \infty$ by (3.1.3), we see from (5.1.1) that all five homomorphisms have finite flat dimension, and that $\sigma_{\mathfrak{q}^*}$ is flat. Two applications of (4.1) yield

$$
\begin{aligned}
\operatorname{cmd}\varphi_{\mathfrak{q}} + \operatorname{cmd}\sigma_{\mathfrak{q}^*} &= \operatorname{cmd}(\sigma_{\mathfrak{q}^*}\varphi_{\mathfrak{q}}) = \operatorname{cmd}(\varphi'_{\mathfrak{q}^*}\varkappa_{\mathfrak{p}'}\rho_{\mathfrak{p}^*}) \\
&\le \operatorname{cmd}\rho_{\mathfrak{p}^*} + \operatorname{cmd}\varkappa_{\mathfrak{p}'} + \operatorname{cmd}\varphi'_{\mathfrak{q}^*}\,.
\end{aligned}
$$

As $\sigma_{\mathfrak{q}^*}$ is flat and $\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big)_{\widetilde{\mathfrak{q}}}$ is its closed fiber, from (3.5) we get

$$
\operatorname{cmd}\sigma_{\mathfrak{q}^*} = \operatorname{cmd}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big)_{\widetilde{\mathfrak{q}}}\,.
$$

Focusing on the right hand side of the inequality, we note first the relations

$$
\operatorname{cmd}\rho_{\mathfrak{p}^*} = \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)_{\mathfrak{p}^*} \le \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)\,,
$$

where the equality comes from (3.5), and the inequality holds by definition. We then record the equalities

$$
\operatorname{cmd}\varkappa_{\mathfrak{p}'} = \operatorname{cmd}\big(k(\mathfrak{p}^*) \otimes_{\widehat{R}} R'\big)_{\mathfrak{p}'} = 0\,,
$$

which follow from (3.5) and (5.5). Finally, we remark that (5.2) and (3.7) yield

$$
\operatorname{cmd}\varphi'_{\mathfrak{q}^*} \le \operatorname{cmd}\varphi' = \operatorname{cmd}\varphi\,.
$$

Summing up, we now have:

$$
\operatorname{cmd}\varphi_{\mathfrak{q}} + \operatorname{cmd}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big)_{\widetilde{\mathfrak{q}}} \le \operatorname{cmd}\varphi + \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)\,.
$$

To finish the proof it remains to take the supremum over $\widetilde{\mathfrak{q}} \in \operatorname{Spec}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big)$. □

6. Cohen–Macaulay defects of a ring homomorphism

The passage from local to global generates two distinct notions of Cohen–Macaulay defect.

(6.1) *Cohen–Macaulay defects of a homomorphism.* For a homomorphism $\varphi : R \to S$ of (not necessarily local) noetherian rings we consider the numbers

$$\operatorname{cmd}\varphi = \sup\left\{ \operatorname{cmd}\varphi_{\mathfrak{n}} \mid \mathfrak{n} \in \operatorname{Max} S \right\},$$

which for a local ring coincides with the one introduced in (3.2), and

$$\operatorname{CMD}\varphi = \sup\left\{ \operatorname{cmd}\varphi_{\mathfrak{q}} \mid \mathfrak{q} \in \operatorname{Spec} S \right\}.$$

Clearly, there is always an inequality $\operatorname{cmd}\varphi \le \operatorname{CMD}\varphi$. Ogoma's example recalled in the preceding section shows that it may be strict in general. On the other hand, equality holds in some important cases.

(6.2) *Homomorphisms essentially of finite type.* If φ is essentially of finite type, then $\operatorname{cmd}\varphi = \operatorname{CMD}\varphi$ by (5.2).

(6.3) *Homomorphisms from Cohen–Macaulay rings.* It follows from (3.1.4) that when R is a Cohen–Macaulay ring, in particular, when φ is the structure homomorphism $\mathbb{Z} \to S$, then $\operatorname{cmd}\varphi = \operatorname{cmd} S$. In particular, $\operatorname{cmd}\varphi = \operatorname{CMD}\varphi$.

The next remark shows that a flat homomorphism φ with $\operatorname{CMD}\varphi \le n$ is precisely what in the terminology of EGA [16, (6.8.1.i)] is called a *homomorphism of codepth at most n.*

(6.4) *Flat homomorphisms.* When $\varphi\colon R \to S$ is flat, (3.5) shows that

$$\operatorname{CMD}\varphi = \sup\left\{ \operatorname{cmd}\big(k(\mathfrak{q}\cap R) \otimes_R S\big) \mid \mathfrak{q} \in \operatorname{Spec} S \right\}.$$

In order to give a "global" version of (5.3) we consider yet another numerical measure of the (non)Cohen–Macaulayness of a noetherian ring.

(6.5) *Formal fibers.* Theorem (5.3) brings into play the the *formal fibers* of the local ring R, that is, the fibers of its completion map $\rho\colon R \to \widehat{R}$. In accordance with the general philosophy of this paper, we introduce a measure of the deviations of the formal fibers of R from being Cohen–Macaulay, by setting

$$\operatorname{fcmd} R = \sup\left\{ \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big) \mid \mathfrak{p} \in \operatorname{Spec} R \right\},$$

and call this number the *formal Cohen–Macaulay defect* of R. Comparison with (6.4) shows that it can alternatively be defined by $\operatorname{fcmd} R = \operatorname{CMD}\rho$.

For a general noetherian ring R we set $\operatorname{fcmd} R = \sup\left\{ \operatorname{fcmd} R_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} R \right\}$.

In the context of our investigation, the significance of this number comes from the fact that it provides an absolute upper bound on the failure of the Cohen–Macaulay defect of a large class of homomorphisms of source R to localize. We say that φ is *locally of finite flat dimension* if for each prime ideal $\mathfrak{q}$ in S the R– (or, equivalently, $R_{\mathfrak{q}\cap R}$)–module $S_{\mathfrak{q}}$ has a finite resolution by flat modules; the same notion results from imposing the condition only on the maximal ideals of S.

(6.6) **Theorem.** *If $\varphi : R \to S$ is locally of finite flat dimension, then*

$$\left.\begin{matrix}\operatorname{CMD}\varphi \\ \operatorname{fcmd} S\end{matrix}\right\} \leq \operatorname{cmd}\varphi + \operatorname{fcmd} R\,.$$

The upper inequality of the theorem is a direct consequence of (5.3), and has a special case important enough to be singled out.

(6.7) *Remark.* We say that a ring R has *Cohen–Macaulay formal fibers* if $\operatorname{fcmd} R = 0$.

This is the case, for instance, when R is a homomorphic image of a Cohen–Macaulay ring Q: the formal fibers of R are then among those of Q, which are Cohen–Macaulay by (5.4). Thus, the next corollary yields a partial generalization of the last assertion of (6.3).

Corollary. *If R has Cohen–Macaulay formal fibers and φ is locally of finite flat dimension, then* $\operatorname{cmd}\varphi = \operatorname{CMD}\varphi$. □

The proof of the lower inequality needs some preparation.

(6.8) **Lemma.** *If $\varphi : R \to S$ is a flat local homomorphism, then*

$$\operatorname{fcmd} R \leq \operatorname{fcmd} S + \operatorname{CMD}\varphi\,.$$

Proof. Let $\rho\colon R \to \widehat{R}$ and $\sigma\colon S \to \widehat{S}$ be the completion maps, and let $\mathfrak{p}^* \in \operatorname{Spec}\widehat{R}$ be given. Pick by faithful flatness $\mathfrak{q}^* \in \operatorname{Spec}\widehat{S}$ lying over $\mathfrak{p}^*$, set $\mathfrak{q} = \mathfrak{q}^* \cap S$ and $\mathfrak{p} = \mathfrak{p}^* \cap R$, and consider the commutative square of local homomorphisms:

$$\begin{array}{ccc} R_{\mathfrak{p}} & \xrightarrow{\varphi_{\mathfrak{q}}} & S_{\mathfrak{q}} \\ {\scriptstyle \rho_{\mathfrak{p}^*}}\downarrow & & \downarrow{\scriptstyle \sigma_{\mathfrak{q}^*}} \\ \widehat{R}_{\mathfrak{p}^*} & \xrightarrow[\widehat{\varphi}_{\mathfrak{q}^*}]{} & \widehat{S}_{\mathfrak{q}^*} \end{array}$$

As all of them are flat, (3.5) yields

$$\operatorname{cmd}\rho_{\mathfrak{p}^*} \leq \operatorname{cmd}\rho_{\mathfrak{p}^*} + \operatorname{cmd}\widehat{\varphi}_{\mathfrak{q}^*} = \operatorname{cmd}\varphi_{\mathfrak{q}} + \operatorname{cmd}\sigma_{\mathfrak{q}^*}\,.$$

By the definitions of $\operatorname{CMD}\varphi$ and $\operatorname{fcmd} S$, this implies $\operatorname{cmd}\rho_{\mathfrak{p}^*} \leq \operatorname{fcmd} S + \operatorname{CMD}\varphi$, and hence $\operatorname{fcmd} R = \operatorname{CMD}\rho \leq \operatorname{fcmd} S + \operatorname{CMD}\varphi$ as required. □

The following result provides a quantitative version of the known fact that if a local ring R has Cohen–Macaulay formal fibers, then so do its localizations, cf. [16, (7.4.5)]

(6.9) **Lemma.** *If* $\mathfrak{p}$ *is a prime ideal in a local ring* R*, then* $\operatorname{fcmd} R_{\mathfrak{p}} \leq \operatorname{fcmd} R$*.*

Proof. Choose, by faithful flatness of the completion map $\rho\colon R \to \widehat{R}$, a prime ideal $\mathfrak{p}^*$ in $\widehat{R}$ lying over $\mathfrak{p}$, and apply (6.8) to the flat local homomorphism $\rho_{\mathfrak{p}^*}\colon R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{p}^*}$ to get

$$\operatorname{fcmd} R_{\mathfrak{p}} \leq \operatorname{fcmd} \widehat{R}_{\mathfrak{p}^*} + \operatorname{CMD} \rho_{\mathfrak{p}^*}\,.$$

The complete local ring $\widehat{R}$ is a homomorphic image of a regular local ring, hence it has Cohen–Macaulay formal fibers, cf. e.g. (6.7). Thus, in the inequality above the first summand on the right hand side is trivial, so it remains to note that, by definition,

$$\operatorname{CMD} \rho_{\mathfrak{p}^*} \leq \operatorname{CMD} \rho = \operatorname{fcmd} R\,. \qquad \square$$

Proof of the lower inequality in (6.6). If φ is local, $\mathfrak{q} \in \operatorname{Spec} S$, and $\mathfrak{p} = \mathfrak{q} \cap R$, then

$$\operatorname{cmd}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big) \leq \operatorname{cmd}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big) + \operatorname{cmd} \varphi \leq \operatorname{fcmd} R + \operatorname{cmd} \varphi$$

by (5.3) and the definition of $\operatorname{fcmd} R$. This gives the desired result in the local case. In general, choose $\mathfrak{n} \in \operatorname{Max} S$, set $\mathfrak{p} = \mathfrak{n} \cap R$, use the local case and then (6.9) to get

$$\operatorname{fcmd} S_{\mathfrak{n}} \leq \operatorname{fcmd} R_{\mathfrak{p}} + \operatorname{cmd} \varphi_{\mathfrak{n}} \leq \operatorname{fcmd} R + \operatorname{cmd} \varphi\,. \qquad \square$$

We finish this section by two results which show that Cohen–Macaulay defects of homomorphisms behave smoothly under flat base change.

(6.10) **Theorem.** *Let* $\tau\colon R \to T$ *be a flat homomorphism of noetherian rings. If* $\varphi\colon R \to S$ *is a homomorphism essentially of finite type and (locally) of finite flat dimension, then the induced homomorphism* $\varphi \otimes_R T\colon T \to S \otimes_R T$ *has the same property, and*

$$\operatorname{CMD}(\varphi \otimes_R T) \leq \operatorname{CMD} \varphi$$

with equality when τ *is faithfully flat.*

Proof. Clearly, $\varphi \otimes_R T$ is essentially of finite type, and thus in particular the ring $S \otimes_R T$ is noetherian.

By an elementary computation, for $i \in \mathbb{Z}$ there is an isomorphism $\operatorname{Tor}_i^T(S \otimes_R T, N) \cong \operatorname{Tor}_i^R(S, N)$. It follows that $\operatorname{fd}(\varphi \otimes_R T) \leq \operatorname{fd} \varphi$; furthermore, for each prime ideal $\mathfrak{s}$ in $S \otimes_R T$ this inequality localizes to $\operatorname{fd}_T T_{\mathfrak{s}} \leq \operatorname{fd}_R S_{\mathfrak{s} \cap S}$. Thus, $\varphi \otimes_R T$ is (locally) of finite flat dimension along with φ.

In the particular case when the homomorphism τ is local and the map φ is surjective, there is an equality $\operatorname{cmd} \varphi = \operatorname{cmd}(\varphi \otimes_R T)$. Indeed, one easily deduces from the faithful flatness of τ that the imperfections $\operatorname{imp}_R S$ and $\operatorname{imp}_T(S \otimes_R T)$ are equal, and (3.6) shows that this is equivalent to our assertion.

In general, we choose a factorization $R \xrightarrow{\xi} P \xrightarrow{\zeta} S$ with P a localization of a ring of polynomials over R, and ζ a surjective homomorphism. Let $\mathfrak{s}$ be an arbitrary prime ideal in $S \otimes_R T$, and set $\mathfrak{r}' = \mathfrak{s} \cap (P \otimes_R T)$; $\mathfrak{r} = \mathfrak{r}' \cap T$; $\mathfrak{q} = \mathfrak{s} \cap S$; $\mathfrak{p}' = \mathfrak{q} \cap P$; $\mathfrak{p} = \mathfrak{p}' \cap R$. There is a commutative diagram of local homomorphisms

$$\begin{array}{ccccc} R_{\mathfrak{p}} & \xrightarrow{\xi_{\mathfrak{p}'}} & P_{\mathfrak{p}'} & \xrightarrow{\zeta_{\mathfrak{q}}} & S_{\mathfrak{q}} \\ {\scriptstyle \tau_{\mathfrak{r}}}\downarrow & & \downarrow{\scriptstyle (P\otimes_R \tau)_{\mathfrak{r}'}} & & \downarrow{\scriptstyle (S\otimes_R \tau)_{\mathfrak{s}}} \\ T_{\mathfrak{r}} & \xrightarrow[(\xi\otimes_R T)_{\mathfrak{r}'}]{} & (P \otimes_R T)_{\mathfrak{r}'} & \xrightarrow[(\zeta\otimes_R T)_{\mathfrak{s}}]{} & (S \otimes_R T)_{\mathfrak{s}} \end{array} .$$

The horizontal homomorphisms in the left hand square are flat with regular closed fibers, while those in the right hand square are surjective. As in the proof of (5.2), it follows that $\operatorname{cmd} \varphi_{\mathfrak{q}} = \operatorname{cmd} \zeta_{\mathfrak{q}}$, and $\operatorname{cmd}(\varphi \otimes_R T)_{\mathfrak{s}} = \operatorname{cmd}(\zeta \otimes_R T)_{\mathfrak{s}}$. The homomorphism $(P \otimes \tau)_{\mathfrak{r}'}$ is flat and the homomorphism $(\zeta \otimes_R T)_{\mathfrak{s}}$ can be naturally identified with $\zeta_{\mathfrak{q}} \otimes_{P_{\mathfrak{p}'}} (P \otimes_R T)_{\mathfrak{r}'}$. Thus, $\operatorname{cmd} \zeta_{\mathfrak{q}} = \operatorname{cmd}(\zeta \otimes_R T)_{\mathfrak{s}}$ by the particular case treated above. The desired inequality $\operatorname{CMD}(\varphi \otimes_R T) \leq \operatorname{CMD} \varphi$ follows.

When τ is faithfully flat so is $S \otimes_R \tau$, hence each $\mathfrak{q} \in \operatorname{Spec} S$ is the contraction of some prime ideal in $S \otimes_R T$, so that the arguments above yield $\operatorname{CMD}(\varphi \otimes_R T) = \operatorname{CMD} \varphi$. □

In the last result we denote by $\mathfrak{j}_T$ the Jacobson radical of a ring T.

(6.11) **Theorem.** *Let $\varphi \colon R \to S$ be a ring homomorphism, and let $\mathfrak{a} \subset R$ and $\mathfrak{b} \subset S$ be ideals such that $\varphi(\mathfrak{a}) \subseteq \mathfrak{b} \neq S$, and let $\varphi^* \colon R^* \to S^*$ be the induced homomorphism of the corresponding ideal-adic completions. The following then hold.*

(a) $\operatorname{fcmd} R^* \leq \operatorname{fcmd} R$, *with equality when* $\mathfrak{a} \subseteq \mathfrak{j}_R$.
(b) $\operatorname{cmd} \varphi^* \leq \operatorname{cmd} \varphi$, *with equality when* $\mathfrak{b} \subseteq \mathfrak{j}_S$.
(c) *When φ is locally of finite flat dimension so is φ^*, and* $\operatorname{CMD} \varphi^* \leq \operatorname{cmd} \varphi^* + \operatorname{fcmd} R^*$.

Proof. We take note of the equality $\operatorname{Max} S^* = \{ \mathfrak{n}S^* \mid \mathfrak{n} \in \operatorname{V}(\mathfrak{b}) \cap \operatorname{Max} S \}$, and write $\widetilde{\rho} \colon R \to R^*$ and $\widetilde{\sigma} \colon S \to S^*$ for the canonical maps, so that $\varphi^* \widetilde{\rho} = \widetilde{\sigma} \varphi$.

When φ is a local homomorphisms, so is φ^*, and their completions can be canonically identified. Two applications of (3.3) show that $\operatorname{cmd} \varphi^* = \operatorname{cmd} \widehat{\varphi} = \operatorname{cmd} \varphi$. Furthermore, it is easily deduced from the faithful flatness of $\widetilde{\rho}$ and $\widetilde{\sigma}$ that $\operatorname{fd} \varphi^* = \operatorname{fd} \varphi$.

In general, let $\mathfrak{n}^*$ be a maximal ideal of S^*. Pick $\mathfrak{n} \in \operatorname{Max} S$ such that $\mathfrak{n} \supseteq \mathfrak{b}$ and $\mathfrak{n}S^* = \mathfrak{n}^*$, remark that $\mathfrak{n} = \mathfrak{n}^* \cap S$, denote by $\mathfrak{m}^*$ the prime ideal $\mathfrak{n}^* \cap R^*$ in R^*, and set $\mathfrak{m} = \mathfrak{m}^* \cap R$. The local homomorphism $(\varphi^*)_{\mathfrak{n}^*} \colon R^*_{\mathfrak{m}^*} \to S^*_{\mathfrak{n}^*}$ is obtained from the local homomorphism $\varphi_{\mathfrak{n}} \colon R_{\mathfrak{m}} \to S_{\mathfrak{n}}$ by completion along the ideals $\mathfrak{a}R_{\mathfrak{m}}$ and $\mathfrak{b}S_{\mathfrak{n}}$. The local case discussed above shows that $(\varphi^*)_{\mathfrak{n}^*}$ and $\varphi_{\mathfrak{n}}$ have equal Cohen–Macaulay defects and flat dimensions. The inequality in (b) follows. If $\mathfrak{b}$ is in the Jacobson radical of S and $\mathfrak{n}^*$ ranges over $\operatorname{Max} S^*$, then $\mathfrak{n}$ ranges over $\operatorname{Max} S$, hence $\operatorname{cmd} \varphi^* = \operatorname{cmd} \varphi$.

Note next that $\widetilde{\rho} = (\operatorname{id}_R)^*$ for the ideal-adic completions with respect to the ideals $(0) \subset R$ and $\mathfrak{a} \subset R$. Applied to this situation, (b) gives $\operatorname{cmd} \widetilde{\rho} \leq \operatorname{cmd} \operatorname{id}_R = 0$. The lower inequality in (6.6) for the flat homomorphism $\widetilde{\rho}$ yields $\operatorname{fcmd} R^* \leq \operatorname{fcmd} R$, which is (a).

The preceding discussion has already shown that φ^* is locally of finite flat dimension along with φ, hence the inequality in (c) is just the upper inequality in (6.6). □

7. Type of a local homomorphism

There is a well known and useful hierarchy among Cohen-Macaulay local rings, based on the number of irreducible ideals in the primary decomposition of some (or, equivalently, any) ideal generated by a system of parameters. The homological interpretation of this invariant is meaningful for arbitrary local rings. The purpose of this section is to introduce the corresponding measure for local homomorphisms, and derive its basic properties.

(7.0) *Type of a local ring.* We define the type of a local ring $(S, \mathfrak{n}, \ell)$ to be its first non-vanishing Bass number, that is

$$\operatorname{type} S = \mu_S^{\operatorname{depth} S} = \operatorname{rank}_\ell \operatorname{Ext}_S^{\operatorname{depth} S}(\ell, S)\,.$$

(Sometimes – but not in this paper – it is the Bass number $\mu_S^{\dim S} = \operatorname{rank}_\ell \operatorname{Ext}_S^{\dim S}(\ell, S)$ which is referred to as the type of S.) It is well known that type can be computed in terms of a presentation of $\widehat{S}$ as quotient of a regular local ring Q:

$$\operatorname{type} S = \operatorname{rank}_\ell \operatorname{Tor}_g^Q(\ell, \widehat{S})\,, \tag{7.0.1}$$

where $g = \operatorname{pd}_Q \widehat{S}$. To see this, note first that $\dim Q - g = \operatorname{depth} \widehat{S} = \operatorname{depth} S$, and use the self-duality of the Koszul resolution of ℓ over Q to obtain the isomorphism

$$\operatorname{Tor}_g^Q(\ell, S) \cong \operatorname{Ext}_Q^{\dim Q - g}(\ell, S) = \operatorname{Ext}_Q^{\operatorname{depth} S}(\ell, S)\,.$$

Choose then in Q a maximal S–regular sequence $\mathbf{x}$ which is also Q–regular, set $\bar{S} = S/(\mathbf{x})$ and $\bar{Q} = Q/(\mathbf{x})$, and use the standard base change isomorphisms

$$\operatorname{Ext}_Q^{\operatorname{depth} S}(\ell, S) \cong \operatorname{Hom}_{\bar{Q}}(\ell, \bar{S}) = \operatorname{Hom}_{\bar{S}}(\ell, \bar{S}) \cong \operatorname{Ext}_S^{\operatorname{depth} S}(\ell, S)\,.$$

For the rest of this section we fix a local homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$.

(7.1) *Type of a local homomorphism.* In order to introduce a notion of type for a local homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ we make the following observation:

Lemma. *If $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen factorization of $\dot{\varphi}$, then for each $i \in \mathbb{Z}$ the $\widehat{S}$–module $\operatorname{Ext}_{R'}^{\operatorname{depth} R' + i}(\widehat{S}, R')$ does not depend on the choice of the factorization.*

Proof. Let $R_1 \xrightarrow{\dot{\varphi}_1} R_1' \xrightarrow{\varphi_1'} \widehat{S}$ also be a Cohen factorizations of $\dot{\varphi}$. By [6, (1.2)] there exist a local ring Q and surjective homomorphisms $Q \to R'$ and $Q \to R_1'$ with kernels generated by Q–regular sequences. Thus, it suffices to consider the case when $R_1' = R'/(x)$ for a non-zero-divisor $x \in \operatorname{Ker} \varphi'$. It is handled by the standard base-change formula

$$\operatorname{Ext}_{R'}^i(\widehat{S}, R') \cong \operatorname{Ext}_{R_1}^{i-1}(\widehat{S}, R_1) \quad \text{for } i \in \mathbb{Z}\,. \qquad \square$$

With $\nu_T N$ standing for the minimal number of generators of a T–module N, we now define the *type* of φ by the formula

$$\operatorname{type}\varphi = \nu_{\widehat{S}} \operatorname{Ext}^f_{R'}(\widehat{S}, R'),$$

where $f = \operatorname{depth} R' - \operatorname{depth} S = -\operatorname{depth}\varphi'$.

By definition, there are equalities

$$\operatorname{type}\varphi = \operatorname{type}\grave{\varphi} = \operatorname{type}\widehat{\varphi}. \tag{7.1.1}$$

When φ is surjective the expression for $\operatorname{type}\varphi$ simplifies to

$$\operatorname{type}\varphi = \nu_S \operatorname{Ext}^f_R(S, R), \tag{7.1.2}$$

where $f = \operatorname{depth} R - \operatorname{depth} S = -\operatorname{depth}\varphi$: simply note that $R \to \widehat{R} \to \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$, and that the $\widehat{S}$–module $\operatorname{Ext}^f_{\widehat{R}}(\widehat{S}, \widehat{R})$ is the $\mathfrak{n}$-adic completion of the S–module $\operatorname{Ext}^f_R(S, R)$.

By definition $\operatorname{type}\varphi$ is a non-negative integer, and it can actually be zero. For instance, if R has $\dim R > \operatorname{depth} R = 0$ and $\mathfrak{p}$ is a minimal prime ideal in R, then the canonical surjection $\varphi\colon R \to R/\mathfrak{p} = S$ has $f = 0 - \operatorname{depth} S < 0$, and hence $\operatorname{Ext}^f_R(S, R) = 0$.

However, in several important cases the type of φ is actually positive. For homomorphisms of finite flat dimension this follows from the next result.

(7.2) **Proposition.** *If* $\operatorname{fd}\varphi$ *is finite and* $R \to R' \xrightarrow{\varphi'} \widehat{S}$ *is a Cohen factorization of* $\grave{\varphi}$*, then*

$$\operatorname{type}\varphi = \operatorname{rank}_\ell \operatorname{Tor}^{R'}_f(\ell, \widehat{S}),$$

where $f = \operatorname{fd}\varphi' = -\operatorname{depth}\varphi' = \operatorname{depth} R' - \operatorname{depth} S$*. In particular,* $\operatorname{type}\varphi \geq 1$.

Proof. The expressions for f come from (3.6). Let $0 \to F_f \xrightarrow{\partial_f} F_{f-1} \to \dots \to F_0 \to 0$ be a minimal free resolution of R'–module $\widehat{S}$. Applying to it the functor $-^* = \operatorname{Hom}_{R'}(-, R')$ we get an exact sequence

$$(F_{f-1})^* \xrightarrow{(\partial_f)^*} (F_f)^* \to \operatorname{Ext}^f_{R'}(\widehat{S}, R') \to 0.$$

In view of the inclusion $\operatorname{Im}(\partial_f)^* \subseteq \mathfrak{m}'(F_f)^*$, where $\mathfrak{m}'$ is the maximal ideal of R', it implies $\operatorname{type}\varphi = \operatorname{rank}_{R'}(F_f)^*$. As $\operatorname{rank}_{R'}(F_f)^* = \operatorname{rank}_{R'} F_f = \operatorname{rank}_\ell \operatorname{Tor}^{R'}_f(\ell, \widehat{S})$, we are done. □

As seen earlier for Cohen–Macaulay defects, cf. (3.5), a numerical invariant of a flat local homomorphism is equal to that of its closed fiber.

(7.3) **Proposition.** *If* $\varphi\colon R \to S$ *is flat, then* $\operatorname{type}\varphi = \operatorname{type}(S/\mathfrak{m}S)$.

Proof. Let $R \to R' \to \widehat{S}$ be a Cohen factorization of $\grave{\varphi}$. It is shown in [4, (2.4.1)] that

$$\operatorname{Tor}_i^{R'}(\ell, \widehat{S}) \cong \operatorname{Tor}_i^{\bar{R}}(\ell, \bar{S}) \quad \text{for } i \in \mathbb{Z}\,,$$

where $\bar{R}' = R'/\mathfrak{m}R'$ and $\bar{S} = \widehat{S}/\mathfrak{m}\widehat{S}$. It follows that $\operatorname{pd}_{R'} \widehat{S} = \operatorname{pd}_{\bar{R}'} \bar{S} =$ (say) f. By applying successively (7.2), the isomorphism for $i = f$, and (7.0.1), we get:

$$\operatorname{type}\varphi = \operatorname{rank}_\ell \operatorname{Tor}_f^{R'}(\ell, \widehat{S}) = \operatorname{rank}_\ell \operatorname{Tor}_f^{\bar{R}'}(\ell, \bar{S}) = \operatorname{type}\bar{S}\,. \qquad \square$$

The type of any homomorphism of finite flat dimension can be expressed in terms of the types of its source and target rings.

(7.4) **Proposition.** *If* $\varphi\colon R \to S$ *is a local homomorphisms of finite flat dimension, then*

$$\operatorname{type}\varphi = \operatorname{type} S / \operatorname{type} R\,.$$

Proof. Consider a Cohen factorization of $\grave{\varphi}$ with surjective homomorphism $\varphi'\colon R' \to \widehat{S}$. Choose a Cohen presentation of the complete local ring R' as a quotient of a regular local ring Q. In the usual change of rings spectral sequence

$${}^2\mathrm{E}_{pq} = \operatorname{Tor}_p^{R'}\big(S, \operatorname{Tor}_q^Q(R', \ell)\big) \Rightarrow \operatorname{Tor}_{p+q}^Q(S, \ell)$$

the space ${}^2\mathrm{E}_{pq}$ vanishes outside the rectangle $0 \le p \le f = \operatorname{pd}_{R'} \widehat{S}$ and $0 \le q \le e = \operatorname{pd}_Q R'$. Thus, in the corner (f, e) the spectral sequence yields an isomorphism

$$\operatorname{Tor}_e^Q(R', \ell) \otimes_\ell \operatorname{Tor}_f^{R'}(S, \ell) \cong \operatorname{Tor}_{f+e}^Q(S, \ell)$$

which translates into an equality

$$\operatorname{rank}_\ell \operatorname{Tor}_e^Q(R', \ell) \cdot \operatorname{rank}_\ell \operatorname{Tor}_f^{R'}(S, \ell) = \operatorname{rank}_\ell \operatorname{Tor}_{e+f}^Q(S, \ell)\,.$$

By (7.2) and (7.0.1) this is precisely the desired formula. $\square$

The multiplicativity of type on compositions of local homomorphisms of finite flat dimension follows immediately:

(7.5) **Corollary.** *If* $\psi\colon Q \to R$ *and* $\varphi\colon R \to S$ *are local homomorphisms of finite flat dimension, then there is an equality*

$$\operatorname{type}\varphi\psi = \operatorname{type}\psi \cdot \operatorname{type}\varphi\,. \qquad \square$$

Another obvious consequence of the proposition is:

(7.6) **Corollary.** *The local structure homomorphism* η_S *has* $\operatorname{type}\eta_S = \operatorname{type} S$. □

This is subsumed in the next result. Unlike all preceding ones on type, it makes no hypothesis on the flat dimension of the homomorphism.

(7.7) **Proposition.** *If R is Gorenstein, then* $\operatorname{type}\varphi = \operatorname{type} S$. *In particular,* $\operatorname{type}\varphi \geq 1$.

Proof. Choose first a Cohen factorization $R \to R' \xrightarrow{\varphi'} \widehat{S}$ of $\dot{\varphi}$, and then a regular local ring Q mapping onto R'. Consider the standard change of rings spectral sequence

$$_2\mathrm{E}^{pq} = \operatorname{Ext}^p_{R'}\big(\widehat{S}, \operatorname{Ext}^q_Q(R', Q)\big) \Rightarrow \operatorname{Ext}^{p+q}_Q(\widehat{S}, Q)$$

and set $c = \dim Q - \dim R'$, $f = \operatorname{depth} R' - \operatorname{depth}\widehat{S}$, and $g = \operatorname{pd}_Q \widehat{S}$. As the ring R' is Gorenstein, we have $\operatorname{Ext}^q_Q(R', Q) = 0$ for $q \neq c$ and $\operatorname{Ext}^c_Q(R', Q) \cong R'$. Thus the spectral sequence collapses to a collection of isomorphisms $\operatorname{Ext}^p_{R'}(\widehat{S}, R') \cong \operatorname{Ext}^{c+p}_Q(\widehat{S}, Q)$ for $p \in \mathbb{Z}$. As $c + f = g$ by the Auslander–Buchsbaum Equality, the isomorphism for $p = f$ yields $\operatorname{type}\varphi = \nu_{\widehat{S}} \operatorname{Ext}^g_Q(\widehat{S}, Q)$. Applying (7.2) to the homomorphism $\xi\colon Q \to \widehat{S}$ we see that the last number is equal to $\operatorname{rank}_\ell \operatorname{Tor}^Q_g(\ell, \widehat{S})$, which is precisely $\operatorname{type} S$ by (7.0.1). □

(7.8) *DGtype of a local homomorphism.* For a local homomorphism φ with $\operatorname{fd}\varphi < \infty$, a number $\operatorname{DGtype}\varphi$ is introduced from a completely different point of view in [8], and called the *DGtype* of φ. In view of [*idem.*, (5.3)] and (7.4) for such φ we have $\operatorname{DGtype}\varphi = \operatorname{type}\varphi$.

(7.9) *Type of a ring.* There is no commonly accepted definition of type for general noetherian rings, and the reason is that the type of a local ring (7.0) is not a local invariant.

For example, let S be the quotient of the polynomial ring $\ell[X, Y, Z]$ over a field ℓ by the ideal (XYZ, Y^2, Z^2), and let x, y, z denote the images of X, Y, Z, respectively. At the maximal ideal $\mathfrak{n} = (x, y, z)$ we have $\operatorname{depth} S_\mathfrak{n} = 0$ and $\big(0 : \mathfrak{n}S_\mathfrak{n}\big) = (yz)S_\mathfrak{n}$, hence $\operatorname{type} S = 1$. On the other hand, at the minimal prime ideal $\mathfrak{q} = (y, z)$, the artinian ring $S_\mathfrak{q} \cong \ell(X)[Y, Z]/(Y^2, YZ, Z^2)$ has $\big(0\colon \mathfrak{q}S_\mathfrak{q}\big) = (y, z)S_\mathfrak{q}$, so that $\operatorname{type} S_\mathfrak{q} = 2$.

In the example there are prime ideals $\mathfrak{q} \subset \mathfrak{n}$ in S with an inequality $\operatorname{type} S_\mathfrak{q} > \operatorname{type} S_\mathfrak{n}$. However, this does not happen if the ring S is (locally) Cohen–Macaulay: the Cohen–Macaulay local rings $S_\mathfrak{q}$ and $S_\mathfrak{n}$ then satisfy $\operatorname{type} S_\mathfrak{q} \leq \operatorname{type} S_\mathfrak{n}$, cf. [18, (6.16)]. Thus, for a Cohen–Macaulay ring S the equality $\operatorname{type} S = \max\big\{\operatorname{type} S_\mathfrak{q} \mid \mathfrak{p} \in \operatorname{Spec} S\big\}$ defines a notion of type which extends the one considered in (7.0) for local rings.

When s is a positive number, we say that a ring S is CM_s if it is Cohen–Macaulay and $\operatorname{type} S_\mathfrak{q} \leq s$ for all $\mathfrak{q} \in \operatorname{Spec} S$. Note that CM_r implies CM_s for $s \geq r$, that the strongest condition CM_1 defines (locally) Gorenstein rings, and that the weakest condition CM_∞ simply means that S is a Cohen–Macaulay ring.

8. Locally Cohen–Macaulay homomorphisms

This section provides a synopsis of the results of the paper, specialized to a class of homomorphisms which contain the locally Gorenstein homomorphisms, cf. [1] and [3], in particular the classically studied locally complete intersection homomorphisms. A remarkable fact about the new class is that it displays precise analogs of the stability and rigidity properties which play such an important role in the use of l.c.i. homomorphisms.

(8.1) *Cohen–Macaulay homomorphisms.* A local homomorphism $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is defined in [6, (3.7)] to be *Cohen–Macaulay at* $\mathfrak{n}$ if it is of finite flat dimension and in some Cohen factorization $R \to R' \to \widehat{S}$ of $\grave{\varphi}$ the R'–module $\widehat{S}$ is perfect. It follows from (3.7) that this property does not depend on the choice of the factorization, and that it is equivalent to the conditions $\operatorname{fd}\varphi < \infty$ and $\operatorname{cmd}\varphi = 0$.

A homomorphism of noetherian rings $\varphi\colon R \to S$ is said to be *Cohen–Macaulay at* a prime ideal $\mathfrak{q}$ in S if the induced local homomorphism $\varphi_{\mathfrak{q}}$ is Cohen–Macaulay at $\mathfrak{q}S_{\mathfrak{q}}$, and it is called *locally Cohen–Macaulay* if it has this property at all $\mathfrak{q} \in \operatorname{Spec} S$. This is the case precisely when φ is locally of finite flat dimension and $\operatorname{CMD}\varphi = 0$.

As for rings, the locally Cohen–Macaulay condition for homomorphisms can be refined by bringing in a consideration of types. When s is a positive number we say that φ is CM_s if φ is locally Cohen–Macaulay and $\operatorname{type}\varphi_{\mathfrak{q}} \le s$ for all $\mathfrak{q} \in \operatorname{Spec} S$. In particular, CM_∞ is equivalent to "locally Cohen–Macaulay."

(8.2) *Gorenstein homomorphisms.* In [1] and [3] we defined a homomorphism $\varphi\colon R \to S$ to be *Gorenstein at* a prime ideal $\mathfrak{q}$ in S when $\operatorname{fd}\varphi_{\mathfrak{q}} < \infty$ and there is an integer d such that the Bass numbers $\mu^i_{R_{\mathfrak{q}\cap R}}$ and $\mu^{i+d}_{S_{\mathfrak{q}}}$ are equal for all $i \in \mathbb{Z}$. The homomorphism φ is *locally Gorenstein* if it is Gorenstein at all prime ideals in S.

It is shown in [6, (3.11)] that a local homomorphism $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is Gorenstein at $\mathfrak{n}$ precisely when the kernel of the surjective homomorphism φ' in a Cohen factorization of $\grave{\varphi}$ is a Gorenstein ideal. In view of (7.2) this is equivalent to φ being Cohen–Macaulay with $\operatorname{type}\varphi = 1$. It follows that φ is locally Gorenstein if and only if it is CM_1.

In the results below we consider various properties of the conditions CM_s for $s \ge 1$. When $s = 1$ they become results on Gorenstein conditions, initially discovered in [3] by an entirely different approach.

From our point of view, a local property of a ring should translate as the corresponding local property of the structure homomorphism $\mathbb{Z} \to S$; this does hold for the CM_s property:

(8.3) **Structure homomorphism.** *The structure homomorphism* $\mathbb{Z} \to S$ *is* CM_s *precisely when the ring* S *is* CM_s*.* Cf. (6.3), (7.6), and (7.9).

In the flat case, Cohen–Macaulay properties of a homomorphism are equivalent to the corresponding properties of its fibers. The following result shows, in particular, that the restriction to the flat case of the notion of locally Cohen–Macaulay homomorphism coincides with the concept of Cohen–Macaulay homomorphism adopted in EGA [16, (6.8.1.i)].

(8.4) **Flat homomorphisms.** *A flat homomorphism* $\varphi\colon R \to S$ *is* CM_s *if and only if all the non-trivial fibers of* φ *are* CM_s*.* Cf. (6.4) and (7.3).

Under appropriate finiteness hypotheses, the Cohen–Macaulay property of a homomorphism can be read off its localizations at maximal ideals:

(8.5) **Homomorphisms essentially of finite type.** *Let* $\varphi\colon R \to S$ *be a homomorphism essentially of finite type, and let* s *be a positive number. If* φ *is Cohen–Macaulay of type at most* s *at all maximal ideals of* S*, then* φ *is* CM_s*.*

Proof. By (5.2) the induced homomorphism $\varphi_{\mathfrak{q}}$ is Cohen–Macaulay.

As φ is essentially of finite type, it factors as $\zeta\xi$, with ξ flat local with regular fibers and ζ surjective. This induces a factorization $\varphi_{\mathfrak{q}} = \zeta_{\mathfrak{p}'}\xi_{\mathfrak{q}}$ with similar properties. It follows from (7.5) and (7.3) that $\operatorname{type}\varphi = \operatorname{type}\zeta$ and $\operatorname{type}\varphi_{\mathfrak{q}} = \operatorname{type}\zeta_{\mathfrak{q}}$.

Thus, it suffices to show that $\operatorname{type}\varphi_{\mathfrak{q}} \leq s$ when φ is surjective. In this case the R–module S is perfect by (3.6), say with $\operatorname{grade}_R S = \operatorname{pd}_R S = f$, and then $\operatorname{Ext}^i_R(S,R) = 0$ for $i \neq f$. All these equalities are preserved when R and S are replaced by their localizations at any prime ideal in R, hence (7.1.2) yields the assertion. □

The three theorems below involve ring homomorphisms $\psi\colon Q \to R$ and $\varphi\colon R \to S$.

(8.6) **Flat descent.** *If φ is faithfully flat and $\varphi\psi$ is CM_t, then ψ and φ are both CM_t.*

Proof. First, note that the property of $\varphi\psi$ to be locally of finite flat dimension is transmitted to ψ by the faithful flatness of φ. Next, conclude by (4.1) that both ψ and φ are locally Cohen–Macaulay. Finally, use (7.5) to derive the bounds on types. □

(8.7) **Composition.** *If ψ is CM_r and φ is CM_s, then $\varphi\psi$ is CM_{rs}.* Cf. (4.1) and (7.5).

(8.8) **Decomposition.** *If ψ and φ are locally of finite flat dimension and $\varphi\psi$ is CM_t, then φ is CM_t and ψ is Cohen–Macaulay of type at most t at each prime ideal of R contracted from S.* Cf. (4.2) and (7.5).

The special cases of the last two results when ψ is the structure homomorphism $\mathbb{Z} \to R$ deserve explicit mention: They give complete information on the transfer of Cohen–Macaulay properties of rings by a homomorphism of finite flat dimension.

(8.9) **Ascent.** *If R is CM_s and φ is CM_r, then S is CM_{rs}.*

(8.10) **Descent.** *If S is CM_s and φ is locally of finite flat dimension, then φ is CM_s and for each $\mathfrak{q} \in \operatorname{Spec} S$ the local ring $R_{\mathfrak{q}\cap R}$ is Cohen–Macaulay of type at most s.*

It follows that a homomorphism of Cohen–Macaulay rings is Cohen–Macaulay precisely when it is locally of finite flat dimension, and that a homomorphism from a regular ring is CM_s precisely when the target is a CM_s ring.

The next result is the precise generalization of part (c) of the Main Theorem of [4], from flat local homomorphisms to arbitrary local homomorphisms of finite flat dimension.

(8.11) **Localization.** *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local homomorphism, let $\mathfrak{q}$ be a prime ideal in S, and assume that the formal fiber of R at $\mathfrak{p} = \mathfrak{q} \cap R$ is Cohen–Macaulay. If φ is Cohen–Macaulay at $\mathfrak{n}$, then it is also Cohen–Macaulay at $\mathfrak{q}$, the formal fiber of S at $\mathfrak{q}$ is Cohen–Macaulay, and there is an inequality*

$$\operatorname{type}\varphi_{\mathfrak{q}} \cdot \operatorname{type}\big(k(\mathfrak{q}) \otimes_S \widehat{S}\big) \leq \operatorname{type}\varphi \cdot \operatorname{type}\big(k(\mathfrak{p}) \otimes_R \widehat{R}\big)\,.$$

Proof. The first two assertions follow from (5.3). The inequality of types is obtained by reusing the construction from the proof of that theorem, with substitutions of appropriate properties of type for those of Cohen–Macaulay defects. □

With a blanket assumption on the formal fibers of R, we get as a consequence a refined solution to Grothendieck's Localization Problem for Cohen–Macaulay properties.

(8.12) **Homomorphisms from rings with Cohen–Macaulay formal fibers.** *Consider a homomorphism $\varphi\colon R \to S$ from a ring R whose formal fibers are CM_r for some $r \geq 1$. If there is an $s \geq 1$ such that φ is Cohen–Macaulay of type at most s at all maximal ideals of S, then φ is CM_{rs} and all formal fibers of S are CM_{rs}.*

In the preceding theorem, the hypothesis on R is essential. Indeed, Ferrand and Raynaud [10, (3.2.i)] have constructed for each $s \geq 2$ a one-dimensional – hence Cohen–Macaulay – local domain R, whose completion $\widehat{R}$ has a prime ideal $\mathfrak{q}$ lying over (0) in R, with $\widehat{R}_{\mathfrak{q}}$ an artinian ring of type s. It follows from (3.5) and (7.3) that the completion homomorphism $\rho\colon R \to \widehat{R}$ and its localization $\rho_{\mathfrak{q}}$ are both Cohen–Macaulay, but $\operatorname{type} \rho_{\mathfrak{q}} = s > 1 = \operatorname{type} \rho$.

In the last two results of this section, the Cohen–Macaulay property is obtained respectively from (6.10) and (6.11), while the (in)equalities of types are derived by mimicking the arguments for the quoted results.

(8.13) **Flat base change.** *Let $\tau\colon R \to T$ be a flat homomorphism of noetherian rings, and let $\varphi\colon R \to S$ be a homomorphism essentially of finite type. If φ is CM_s, then so is the induced homomorphism $\varphi \otimes_R T\colon T \to S \otimes_R T$. If, furthermore, τ is faithfully flat, then the converse holds as well.*

(8.14) **Completion.** *Let $\varphi\colon R \to S$ be a homomorphism, let $\mathfrak{a} \subset R$ and $\mathfrak{b} \subset S$ be ideals such that $\varphi(\mathfrak{a}) \subseteq \mathfrak{b} \neq S$, and let $\varphi^*\colon R^* \to S^*$ be the induced homomorphism of the corresponding ideal-adic completions. If the formal fibers of R^* are CM_r and φ is CM_s, then φ^* is CM_{rs}.*

Appendix. Codimension conjectures

Here we examine the obstructions for Cohen–Macaulay defects to be additive on compositions of local homomorphisms of finite flat dimension, in the sense that for such maps the inequality in (4.1) becomes an equality.

One of these obstructions involves the "defects" of chains of prime ideals in varieties defined by ideals of finite projective dimension. More generally, for a finite module M over a local ring R and a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R M$, we consider the integer

$$\operatorname{catd}_R^M(\mathfrak{p}) = \dim_R M - \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \dim(R/\mathfrak{p})$$

which is alway non-negative, cf. (2.3.1), and set $\operatorname{catd}_R(\mathfrak{p}) = \operatorname{catd}_R^R(\mathfrak{p})$.

If R is equidimensional and catenary, then $\operatorname{catd}_R(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. The converse is proved by Ratliff [29, (2.2)] for domains, cf. also [23, (31.4)], and extended to general local rings by McAdam and Ratliff [24, Proposition 7]. Thus, the integer $\sup\{\operatorname{catd}_R(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}$ measures the failure of R to be equidimensional and catenary.

The relevance of chain defects to local homomorphisms is seen from the next result.

(A.1) **Theorem.** *The following statements are equivalent.*

(i) $\dim R/\mathfrak{a} + \operatorname{grade}_R R/\mathfrak{a} = \dim R$ *for any ideal $\mathfrak{a}$ in a local ring R with* $\operatorname{pd}_R \mathfrak{a} < \infty$.

(ii) $\operatorname{cmd} R + \operatorname{cmd}\varphi = \operatorname{cmd} S$ *for all local homomorphisms* $\varphi\colon R \to S$ *with* $\operatorname{fd}\varphi < \infty$.
(iii) $\operatorname{catd}_R(\mathfrak{q} \cap R) \le \operatorname{catd}_S(\mathfrak{q})$ *for all local homomorphisms* $\varphi\colon R \to S$ *with* $\operatorname{fd}\varphi < \infty$ *and all* $\mathfrak{q} \in \operatorname{Spec} S$.

The validity of (i) is the restriction to cyclic modules of a conjecture of M. Auslander, cf. [27, (II.0.8)]. It is discussed, in the context of other "homological conjectures", in [27], [19], [20], and most recently in [33]. Its general form asserts that equality always holds in condition (i) of the next result, whose part (iii) uses the imperfection of M introduced in (2.6), and whose condition (v) should be compared to the generally valid inequality (2.1.a).

(A.2) **Proposition.** *For a finite module* $M \neq 0$ *of finite projective dimension over a local ring* R *the following conditions are equivalent.*

(i) $$\dim R = \operatorname{grade}_R M + \dim_R M\,.$$
(ii) $$\dim R = \operatorname{codim}_R M + \dim_R M\,.$$
(iii) $$\operatorname{cmd} R = \operatorname{cmd}_R M - \operatorname{imp}_R M\,.$$
(iv) $$\operatorname{catd}_R(\mathfrak{p}) \le \operatorname{catd}_R^M(\mathfrak{p}) \quad \textit{for all} \quad \mathfrak{p} \in \operatorname{Supp}_R M\,.$$

They hold if and only if the next statement is true:

(v) *If* R *is a homomorphic image of a regular local ring* Q, *and* $M \neq 0$ *is a finite* R*–module of finite projective dimension, then* $\operatorname{codim}_R M + \operatorname{codim}_Q R = \operatorname{codim}_Q M$.

In condition (iv) Auslander's homological conjecture is recast entirely as a property of chains of prime ideals in $\operatorname{Supp}_R M$. Thus, we obtain the following result, initially proved in [11, (5.5)] in the equicharacteristic case.

(A.3) **Corollary.** *If* R *is a catenary and equidimensional local ring, then for any finite* R*–module* $M \neq 0$ *of finite projective dimension there is an equality*

$$\dim R = \operatorname{codim}_R M + \dim_R M\,. \qquad \square$$

The proof of the proposition uses a curious property of chain defects.

(A.4) **Lemma.** *There is an inequality:*

$$\dim R - \dim_R M - \operatorname{grade}_R M \ge \sup\left\{ \operatorname{catd}_R(\mathfrak{p}) - \operatorname{catd}_R^M(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R M \right\},$$

and equality holds if $\operatorname{pd}_R M$ *is finite.*

Proof. For $\mathfrak{p} \in \operatorname{Supp}_R M$ write the non-negative integer $\dim R - \dim_R M - \operatorname{grade}_R M$, denoted $\gamma_R M$ in (2.3), in the form

$$\gamma_R M = \operatorname{catd}_R(\mathfrak{p}) - \operatorname{catd}_R^M(\mathfrak{p}) + \gamma_{R_\mathfrak{p}} M_\mathfrak{p} + (\operatorname{grade}_{R_\mathfrak{p}} M_\mathfrak{p} - \operatorname{grade}_R M)\,.$$

The obvious inequality $\operatorname{grade}_R M \le \operatorname{grade}_{R_\mathfrak{p}} M_\mathfrak{p}$ implies the desired one.

When $\operatorname{pd}_R M$ is finite, choose by (2.5) a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R M$ with $\dim R_{\mathfrak{p}} = \operatorname{grade}_R M$ and $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$ and note that

$$\operatorname{catd}_R(\mathfrak{p}) - \operatorname{catd}_R^M(\mathfrak{p}) = \operatorname{catd}_R(\mathfrak{p}) - \operatorname{catd}_R^M(\mathfrak{p}) + (\dim R_{\mathfrak{p}} - \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{grade}_R M)$$

is equal to $\gamma_R M$, and thus is non-negative. □

Proof of (A.2). The equivalence of (i) with (ii), respectively (iii), respectively (iv) is due to (2.5), respectively to the Auslander–Buchsbaum Equality, respectively to (A.4).

If (ii) holds, then applying it to the R–module M and to the Q–module R one obtains the equality in (v). For the converse, choose a regular local ring Q mapping onto the completion $\widehat{R}$ of R. Using (v) and the Cohen–Macaulayness of Q one gets

$$\begin{aligned}\operatorname{codim}_R M &= \operatorname{codim}_{\widehat{R}} \widehat{M} = \operatorname{codim}_Q \widehat{M} - \operatorname{codim}_Q \widehat{R} \\ &= (\dim Q - \dim_Q \widehat{M}) - (\dim Q - \dim_Q \widehat{R}) \\ &= \dim R - \dim_R M\,. \end{aligned}$$ □

Before proving (A.1), we show that its condition (iii) does hold when $\operatorname{fd}\varphi = 0$: this is a quantitative sharpening of the known result that the property of being equidimensional and catenary descends by flat local homomorphisms, cf. [16, (7.1.3)] or [23, (31.5)].

(A.5) **Proposition.** *If $\varphi\colon R \to R'$ is a flat local homomorphism, $\mathfrak{p}'$ is a prime ideal in R' and $\mathfrak{p} = \mathfrak{p}' \cap R$ is its inverse image in R, then:*

$$\operatorname{catd}_{R'}(\mathfrak{p}') = \operatorname{catd}_R(\mathfrak{p}) + \operatorname{catd}_{R'/\mathfrak{p}R'}(\mathfrak{p}'/\mathfrak{p}R')\,.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Note that the flatness of φ implies that the canonical homomorphisms $R_{\mathfrak{p}} \to {R'}_{\mathfrak{p}'}$ and $R/\mathfrak{p} \to R'/\mathfrak{p}R'$ are flat as well, hence

$$\begin{aligned}\dim R' &= \dim R + \dim(R'/\mathfrak{m}R')\,. \\ \dim {R'}_{\mathfrak{p}'} &= \dim R_{\mathfrak{p}} + \dim({R'}_{\mathfrak{p}'}/\mathfrak{p}{R'}_{\mathfrak{p}'})\,. \\ \dim(R'/\mathfrak{p}R') &= \dim(R/\mathfrak{p}) + \dim(R'/\mathfrak{m}R')\,. \end{aligned}$$

They easily combine to give the result, due to the identifications $R'/\mathfrak{p}' = (R'/\mathfrak{p}R')/(\mathfrak{p}'/\mathfrak{p}R')$ and $R'_{\mathfrak{p}'}/\mathfrak{p}R'_{\mathfrak{p}'} = (R'/\mathfrak{p}R')_{\mathfrak{p}'/\mathfrak{p}R'}$. □

Proof of (A.1). Note that (i) follows from (ii) by (3.6), and from (iii) due to (A.2). We assume next that (i) holds and consider a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of $\dot{\varphi}$.

To obtain (ii), note that by the definitions in (3.2) it suffices to prove that $\dim R' - \operatorname{ht}\operatorname{Ker}\varphi' = \dim \widehat{S}$. As the ideal $\operatorname{Ker}\varphi' \subset R'$ has finite projective dimension by (3.1.3), and hence its height and grade coincide by (2.5), the desired equality is provided by (i).

To get (iii), start with a $\mathfrak{q} \in \operatorname{Spec} S$ and choose by faithful flatness $\mathfrak{q}^* \in \operatorname{Spec} \widehat{S}$ lying over it and having $\dim \widehat{S}/\mathfrak{q}^* = \dim \widehat{S}/\mathfrak{q}\widehat{S}$. Note that $\dim \widehat{S}/\mathfrak{q}\widehat{S} = \dim S/\mathfrak{q}S$, and that the flatness of the induced homomorphism $S_\mathfrak{q} \to \widehat{S}_{\mathfrak{q}^*}$ yields $\dim S_\mathfrak{q} \leq \dim \widehat{S}_{\mathfrak{q}^*}$. Thus, we get

$$\operatorname{catd}_S(\mathfrak{q}) = \dim S - \dim(S/\mathfrak{q}) - \dim S_\mathfrak{q} \geq \dim \widehat{S} - \dim(\widehat{S}/\mathfrak{q}^*) - \dim \widehat{S}_{\mathfrak{q}^*} = \operatorname{catd}_{\widehat{S}}(\mathfrak{q}^*)\,.$$

As $\widehat{S}$ is a homomorphic image of R', there is an equality $\operatorname{catd}_{\widehat{S}}(\mathfrak{q}^*) = \operatorname{catd}^{\widehat{S}}_{R'}(\mathfrak{p}')$, where $\mathfrak{p}' = \mathfrak{q}^* \cap R'$. Because the projective dimension of the R'–module $\widehat{S}$ is finite by (3.1.3), we can apply (A.2) to obtain an inequality $\operatorname{catd}^{\widehat{S}}_{R'}(\mathfrak{p}') \geq \operatorname{catd}_{R'}(\mathfrak{p}')$. By (A.5) we now have

$$\operatorname{catd}_{R'}(\mathfrak{p}') \geq \operatorname{catd}_R(\mathfrak{p}' \cap R) = \operatorname{catd}_R(\mathfrak{q} \cap R)\,,$$

and this finishes the proof of the theorem. □

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