A NEW COMBINATORIAL FORMULA FOR CLUSTER MONOMIALS OF EQUIORIENTED TYPE A QUIVERS


Abstract. Cluster algebras were first introduced by S. Fomin and A. Zelevinsky in 2001. Since then, an important question has been how to explicitly construct good bases in them, that is, bases having positive structure constants, containing the cluster monomials, etc. There currently exists a formula for the cluster variables of cluster algebras associated to polygons in terms of \( T \)-paths, which has been modified and generalized to cluster algebras coming from surfaces. However, this formula does not seem to directly generalize to arbitrary cluster algebras. In this paper, we give a new (and more compact) combinatorial formula for all cluster monomials of the cluster algebra associated to any equioriented type \( A \) quiver, in terms of compatible subpaths on maximal Dyck paths. This formula has the potential to generalize to cluster algebras beyond the ones coming strictly from surfaces.

1. INTRODUCTION

A (skew-symmetric) cluster algebra \( \mathcal{A} \) is a subalgebra of a rational function field with a distinguished set of variables (cluster variables), grouped into overlapping subsets (clusters), and defined by a recursive procedure (mutation) on quivers. A cluster monomial is a monomial in the elements of any cluster. It is expected that each cluster algebra admits a “good” basis, that is, a basis having positive structure constants, containing the cluster monomials, etc. Caldero and Keller [1] showed that the cluster monomials form a basis of the cluster algebra if and only if there is only a finite number of cluster variables. Recently Cerulli Irelli, Keller, Labardini-Fragoso and Plamondon [2] proved that the cluster monomials are linearly independent for all cluster algebras associated to quivers.

Given any quiver \( Q \) without loops and 2-cycles, a unique (coefficient-free) cluster algebra \( \mathcal{A}(Q) \) (up to isomorphism) can be defined. In this paper we deal with the quiver

\[
Q = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow ... \longrightarrow n .
\]

It is well known that every quiver whose underlying graph is the Dynkin diagram of type \( A_n \) can be obtained from this quiver by a finite sequence of mutations. In order to make some arguments simpler, we actually consider an extended quiver

\[
Q' = 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow ... \longrightarrow n \longrightarrow n + 1 ,
\]

Date: September 28, 2014.
2010 Mathematics Subject Classification. 13F60.
Key words and phrases. cluster algebra.
The email address of the corresponding author: klee@math.wayne.edu.
where 0 and \(n + 1\) are frozen vertices.

In [8], Schiffler (independently Carroll-Price and Fomin-Zelevinsky) obtained a formula for the cluster variables of \(A(Q)\) in terms of \(T\)-paths. This formula has been generalized to cluster algebras coming from surfaces [6, 7, 9, 10]. However it does not seem to directly generalize to arbitrary cluster algebras.

In this paper we give a new (and more compact) combinatorial formula for all cluster monomials of \(A(Q)\) in terms of compatible subpaths on maximal Dyck paths. In [5] the cluster monomials of rank 2 quivers, which do not necessarily come from surfaces, are described in terms of Dyck paths. We hope that Dyck path formulae may generalize to cluster algebras beyond the ones coming strictly from surfaces.

This article is organized as follows. In Section 2, we introduce our main result. Section 3 gives a proof of the formula for cluster variables using results from [8]. In Section 4 we prove the main result. Finally, in Section 5, we prove the equivalence our definition to a special case of the more general definition in [5, Definition 1.10].

Acknowledgments. We are grateful to Li Li and Dylan Rupel for their valuable comments.

2. Main result

Let \((a_1, a_2)\) be a pair of nonnegative integers. Let \(m = \min(a_1, a_2)\). A maximal Dyck path of type \(a_1 \times a_2\), denoted by \(D = D^{a_1 \times a_2}\), is a lattice path from \((0, 0)\) to \((a_1, a_2)\) that is as close as possible to the diagonal joining \((0, 0)\) and \((a_1, a_2)\), but never goes above it. A corner is a subpath consisting of a horizontal edge followed by a vertical edge. Let \(D_1 = \{u_1, \ldots, u_{a_1}\}\), where \(u_i\) is the horizontal edge of the \(i\)-th corner for \(i \in [1, m]\) and \(u_{m+i}\) is the \(i\)-th of the remaining horizontal ones for \(i \in [1, a_1 - m]\). Let \(D_2 = \{v_1, \ldots, v_{a_2}\}\), where \(v_i\) is the vertical edge of the \(i\)-th corner for \(i \in [1, m]\) and \(v_{m+i}\) is the \(i\)-th of the remaining vertical ones for \(i \in [1, a_2 - m]\).

![Figure 1. A maximal Dyck path.](image)

Definition 2.1. Let \(S_1 \subseteq D_1\) and \(S_2 \subseteq D_2\). We say that \(S_1\) and \(S_2\) are locally compatible with respect to \(D\) if and only if no horizontal edge in \(S_1\) is the immediate predecessor of any vertical edge in \(S_2\) on \(D\).

Remark 2.2. Notice that we have: \(|\mathcal{P}(D_1) \times \mathcal{P}(D_2)|\) possible pairs for \((S_1, S_2)\), where \(\mathcal{P}(D_1)\) denotes the power set of \(D_1\) and \(\mathcal{P}(D_2)\) denotes the power set of \(D_2\). Further, when either \(S_1\) or \(S_2 = \emptyset\), any arbitrary choice of the other will yield local compatibility by our above definition.
For an integer $x$, denote $\max(0, x)$ by $[x]_+$. 

**Definition 2.3.** Let $(a_1, a_2, ..., a_n)$ be an $n$-tuple of integers. Let $a_0 = a_{n+1} = 0$. Denote $D^{[a_1]_+ \times [a_{n+1}]_+}$ by $D^{(i)}$. Let $S_{i,j} \subseteq D^{(i)}$ for $i \in [0, n], j \in [1, 2]$. We say that the collection of $S_{i,j}$ is **globally compatible** if and only if

1. $S_{i,1}$ and $S_{i,2}$ are **locally compatible** with respect to $D^{(i)}$ for all $i \in [1, n-1]$;
2. $v_k(i) \in S_{i,2}$ if and only if $u_k(i+1) \notin S_{i+1,1}$ for all $i \in [0, n-1]$ and all $k$.

**Definition 2.4.** For each $a = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$, we define a Laurent polynomial

$$\tilde{x}[a] := \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{i=0}^{n} x_i^{[S_{i,2}]} x_i^{[S_{i,1}]} ,$$

where the sum runs over all globally compatible collections (abbreviated GCCs) on $D^{(0)} \times \ldots \times D^{(n)}$.

Let

$$Q = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots \longrightarrow n$$

and

$$Q' = 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots \longrightarrow n \longrightarrow n + 1 .$$

The following is our main result.

**Theorem 2.5.** $\tilde{x}[a]$ are cluster monomials of $A(Q, Q')$. Conversely, every cluster monomial is of the form $\tilde{x}[a]$. In particular, if $x_0 = x_{n+1} = 1$, then $\tilde{x}[a]$ form the (dual-)canonical basis for $A(Q)$.

### 3. Cluster Variables

In this section we show that every cluster variable is of the form $\tilde{x}[a]$ using results from [8]. We explicitly describe all such $\tilde{x}[a]$.

Let $n$ be a positive integer, and let $P$ be a regular polygon with $n + 3$ vertices. Let $\{v_0, \ldots, v_{n+2}\}$ be the set of vertices of $P$ where $v_0$ is a vertex of $P$ and $v_i$ is the $i$-th vertex counterclockwise from $v_0$. A **diagonal** of $P$ is a line segment connecting two non-adjacent vertices. Two diagonals are said to be **crossing** if they intersect in the interior of $P$. A **triangulation** of $P$ is a maximal set of non-crossing diagonals together with the boundary edges of $P$.

Our initial triangulation of $P$ will consist of the set $T = \{T_0, \ldots, T_{n+1}, T_{n+2}, \ldots, T_{2n+2}\}$, where $T_i$ is the edge/diagonal that connects $v_i$ and $v_{n+2}$, for $0 \leq i \leq n + 1$ and $T_j$ are the remaining edges of $P$ for $n + 2 \leq j \leq 2n + 2$. See Figure 2. It is known that triangulations of the polygon $P$ are in bijection with clusters of rank $n$. Therefore $\{T_1, \ldots, T_n\}$ are in bijection with the cluster variables $\{x_1, \ldots, x_n\}$. It is known in [3] that the cluster monomials associated to the initial triangulation are equal to those associated to the quiver $Q'$.

We recall the definition of a $T$-path from [8]. Let $w$ and $v$ be two non-adjacent vertices on the boundary and let $M_{w,v}$ be the diagonal that connects $w$ and $v$. 


Definition 3.1. [8] A $T$-path $\alpha$ from $w$ to $v$ is a sequence

$$\alpha = w_0 \xrightarrow{T_{i_1}} w_1 \xrightarrow{T_{i_2}} w_2 \xrightarrow{T_{i_3}} \cdots \xrightarrow{T_{i_{\ell(\alpha)}}} w_{\ell(\alpha)}$$

such that

1. $w = w_0, w_1, ..., w_{\ell(\alpha)} = v$ are vertices of $P$.
2. $i_k \in \{0, 1, ..., 2n + 2\}$ such that $T_{i_k}$ connects the vertices $w_{k-1}$ and $w_k$ for each $k = 1, 2, ..., \ell(\alpha)$.
3. $i_j \neq i_k$ if $j \neq k$.
4. $\ell(\alpha)$ is odd.
5. $T_{i_k}$ crosses $M_{w,v}$ if $k$ is even.
6. If $j < k$ and both $T_{i_j}$ and $T_{i_k}$ cross $M_{w,v}$ then the intersection point of $T_{i_j}$ and $M_{w,v}$ is closer to the vertex $w$.

For any $T$-path $\alpha$, let

$$x(\alpha) = \prod_{k:\text{odd}} x_{i_k} \prod_{k:\text{even}} x_{i_k}^{-1},$$

where $x_{n+2} = ... = x_{2n+2} = 1$.

Theorem 3.2. [8] Let $M = M_{w,v}$ and let $x_M$ be the corresponding cluster variable. Then

$$x_M = \sum_{\alpha : T\text{-path from } w \text{ to } v} x(\alpha).$$

We want to describe denominators of cluster variables.

Definition 3.3. Let $CV \subset \mathbb{Z}^n$ be the set of all $a = (a_1, ..., a_n)$ such that

1. $a_i \in \{-1, 0, 1\} \forall i \in [1, n]$;
2. If $a_i = 1$ and $a_{i+k} = 1$ then $a_{i+j} = 1$ for all $j \in [1, k - 1]$;
3. If $a_i = -1$ then $a_j = 0$ for all $j \neq i$.

If $a \in CV$ and $a_i = -1$ for some $i$ then $\tilde{x}[a]$ is equal to $x_i$. 
Throughout the remainder of this section, fix $k \in [0, n - 1]$, $m \in [1, n - k]$, and suppose that $a$ is the element of $CV$ such that
\begin{equation*}
    a_i = \begin{cases} 
        1, & \text{if } i \in [k + 1, k + m]; \\
        0, & \text{otherwise}.
    \end{cases}
\end{equation*}

**Definition 3.4.** Let $M = M_{v_k,v_{k+m+1}}$ be the diagonal that connects the vertices $v_k$ and $v_{k+m+1}$. Then \( \{x_{k+1}, x_{k+2}, \ldots, x_{k+m}\} \) represents the set of variables corresponding to the diagonals which $M$ crosses. Let $\mathcal{P}$ denote the set of all $T$-paths from $v_k$ to $v_{k+m+1}$, and $\mathcal{W}$ denote the set of all GCCs on $D^{(k)} \times \cdots \times D^{(k+m)}$.

**Definition 3.5.** For each $\beta \in \mathcal{W}$, let $\beta = (S^{(k)}, \ldots, S^{(k+m)})$ where $S^{(i)}$ is the subpath of $D^{(i)}$ consisting of the elements of the sets $S_i,1$ and $S_i,2$. Denote $S^{(i)}$ by
\begin{enumerate}
    \item $|S_i,1|, |S_i,2|$ for $i \in [k + 1, k + m - 1]$;
    \item $(\emptyset, |S_i,2|)$ for $i = k$;
    \item $(|S_i,1|, \emptyset)$ for $i = k + m$.
\end{enumerate}

**Lemma 3.6.** The following are the only possible $T$-paths:
\begin{align*}
    \alpha_k &= v_k \xrightarrow{T_{n+k+2}} v_{k+1} \xrightarrow{T_{k+1}} v_{n+2} \xrightarrow{T_{k+m+1}} v_{k+m+1}; \\
    \alpha_i &= v_k \xrightarrow{T_k} v_{n+2} \xrightarrow{T_i} v_{i+1} \xrightarrow{T_{i+1}} v_{n+2} \xrightarrow{T_{k+m+1}} v_{k+m+1}, \quad \text{for } i \in [k + 1, k + m - 1]; \\
    \alpha_{k+m} &= v_k \xrightarrow{T_k} v_{n+2} \xrightarrow{T_{k+m}} v_{k+m} \xrightarrow{T_{n+k+m+2}} v_{k+m+1}.
\end{align*}

**Proof.** Using [8, Lemma 2.2], the first segment in the $T$-path should be either $T_k$ or the edge towards $v_{k+1}$. If the $T$-path travels from $v_k$ along $T_k$, then by the way triangulations were defined, the $T$-path necessarily travels toward $v_{n+2}$. Since we require that $T_{i_k}$ crosses $M$ for $k$ even, the $T$-path must next travel along a diagonal towards some $v_i$ where $i \in [k + 1, k + m]$. Then the only possible forward direction is to take the edge to $v_{i+1}$. If $i = k + m$, the $T$-path is completed; otherwise the $T$-path should travel along the diagonal $T_{i+1}$, which again returns the $T$-path to $v_{n+2}$. Now, the edge needs to choose its $i$-th segment where $i$ is odd. If the $T$-path travels along a diagonal which $M$ crosses, it is impossible for it to travel along another diagonal which $M$ crosses at the next even step. Thus the only edge available is $T_{k+m+1}$, completing the $T$-path.

Instead, if the $T$-path begins with the edge towards $v_{k+1}$, it must next take the diagonal $T_{k+1}$ toward the vertex $v_{n+2}$. Again, the $T$-path cannot travel along an edge which $M$ crosses at this odd step and it should end with the edge $v_{k+m+1}$. \hfill \Box

**Corollary 3.7.**
\begin{align*}
    x(\alpha_k) &= \frac{x_{k+m+1}}{x_{k+1}}, \\
    x(\alpha_i) &= \frac{x_k x_{k+m+1}}{x_i x_{i+1}}, \quad \text{for } i \in [k + 1, k + m - 1]; \\
    x(\alpha_{k+m}) &= \frac{x_k}{x_{k+m}}.
\end{align*}
Lemma 3.8. Let \( \phi : \mathcal{P} \to \mathcal{W} \) be defined by \( \phi(\alpha_i) = (\beta_i) = (S^{(k)} \ldots S^{(k+m)}) \) where

For \( i = k \)

\[
S^{(j)} = \begin{cases} 
(\emptyset, 0) & \text{for } j = k \\
(1, 0) & \text{for } k + 1 \leq j \leq k + m - 1 \\
(1, \emptyset) & \text{for } j = k + m 
\end{cases}
\]

For \( k + 1 \leq i \leq k + m - 1 \)

\[
S^{(j)} = \begin{cases} 
(\emptyset, 1) & \text{for } j = k \\
(0, 1) & \text{for } k + 1 \leq j \leq i - 1 \\
(0, 0) & \text{for } j = i \\
(1, 0) & \text{for } i + 1 \leq j \leq k + m - 1 \\
(1, \emptyset) & \text{for } j = k + m 
\end{cases}
\]

For \( i = k + m \)

\[
S^{(j)} = \begin{cases} 
(\emptyset, 1) & \text{for } j = k \\
(0, 1) & \text{for } k + 1 \leq j \leq k + m - 1 \\
(0, \emptyset) & \text{for } j = k + m 
\end{cases}
\]

Thus \( \beta_i \) is a globally compatible collection for every \( i \in [k, k + m] \).

Proof. Fix \( i \). It is clear from the construction that for every \( j \in [k + 1, k + m - 1] \), \( S^{(j)} \neq (1, 1) \). Thus \( S_{j,1} \) and \( S_{j,2} \) are locally compatible with respect to \( D^{(j)} \). Also, for \( j \in [k + 1, k + m - 2] \), \( |S_{j,2}| = 1 \) if and only if \( |S_{j+1,1}| \neq 1 \). Thus to show global compatibility we only need to show:

1. \( S^{(k)} = (\emptyset, 1) \) if and only if \( S^{(k+1)} \neq (1, 0) \)
2. \( S^{(k+m-1)} = (0, 1) \) if and only if \( S^{(k+m)} \neq (1, \emptyset) \)

For (1), \( S^{(k+1)} = (1, 0) \) if and only if \( i + 1 \leq k + 1 \leq k + m - 1 \) and \( i \neq k + m \). This is equivalent to \( i = k \), since \( i \geq k \). So, \( S^{(k+1)} = (1, 0) \) iff \( i = k \). But by the construction of \( \phi \), \( S^{(k)} = (\emptyset, 1) \) if and only if \( i \neq k \). Thus \( S^{(k)} = (\emptyset, 1) \) if and only if \( S^{(k+1)} \neq (1, 0) \)

For (2), \( S^{(k+m-1)} = (0, 1) \) iff \( k + 1 \leq k + m - 1 \leq i - 1 \) and \( i \neq k \). Which is equivalent to \( i = k + m \) as \( i \leq k + m \). So, \( S^{(k+m-1)} = (1, 0) \) iff \( i = k + m \). But by the construction of \( \phi \), \( S^{(k+m)} = (1, \emptyset) \) if and only if \( i \neq k + m \). Thus \( S^{(k+m-1)} = (0, 1) \) if and only if \( S^{(k+m)} \neq (1, \emptyset) \)

\[\blacksquare\]

Lemma 3.9. For every \( \beta \in \mathcal{W} \) there exists a unique \( \alpha_i \in \mathcal{P} \) such that \( \beta = \phi(\alpha_i) \).
Proof. Let $\beta = (S^{(k)}, \ldots, S^{(k+m)}) \in W$.

Case 1: Suppose $S_{(k)} = (\emptyset, 0)$. The global compatibility of $\beta$ implies that $S^{(j)} = (1, 0)$ for every $j \in [k + 1, k + m - 1]$ and $S^{(k+m)} = (1, \emptyset)$. It follows from the construction in Lemma 3.8 that $\beta = \phi(\alpha_k)$.

Case 2: Suppose $S^{(k)} = (\emptyset, 1)$.

Case 2.1: Assume that there exists $i \in [k + 1, k + m - 1]$ such that $S^{(i)} = (0, 0)$. Then the global compatibility of $\beta$ implies that $S^{(j)} = (0, 1)$ for $k + 1 \leq j \leq i - 1$, $S^{(j)} = (1, 0)$ for $i + 1 \leq j \leq k + m - 1$, and $S^{(k+m)} = (1, \emptyset)$. It follows that $\beta = \phi(\alpha_i)$.

Case 2.2: Assume that $S^{(i)} \neq (0, 0)$ for any $i \in [k + 1, k + m - 1]$. Then the global compatibility implies that $S^{(j)} = (0, 1)$ for $k + 1 \leq j \leq k + m - 1$ and $S^{(k+m)} = (0, \emptyset)$. It follows that $\beta = \phi(\alpha_{k+m})$. □

Lemma 3.10. For every $\beta_i = \phi(\alpha_i) \in W$, denote by $Y(\beta_i)$ the Laurent monomial

$$\left( \prod_{u=k+1}^{k+m} x_{u}^{-a_{u}} \right) \cdot \prod_{u=k}^{k+m} x_{u}^{\lfloor S_{u,2} \rfloor} x_{u+1}^{\lfloor S_{u,1} \rfloor},$$

where $\{S_{u,j}\}$ is the globally compatible collection $\beta_i$.

Then $x_M = \sum_{\beta_i \in W} Y(\beta_i)$.

Proof. Using the construction in Lemma 3.8, for $i = k$,

$$Y(\beta_k) = \left( \prod_{u=k+1}^{k+m} x_{u}^{-a_{u}} \right) x_{k+2} \cdots x_{k+m+1} = \frac{x_{k+m+1}}{x_{k+1}}.$$

For $i \in [k + 1, k + m - 1]$,

$$Y(\beta_i) = \left( \prod_{u=k+1}^{k+m} x_{u}^{-a_{u}} \right) x_{k} \cdots x_{i-1} \cdot x_{i+2} \cdots x_{k+m+1} = \frac{x_{k} x_{k+m+1}}{x_{i} x_{i+1}}.$$

For $i = k + m$,

$$Y(\beta_{k+m}) = \left( \prod_{u=k+1}^{k+m} x_{u}^{-a_{u}} \right) x_{k} \cdots x_{k+m-1} = \frac{x_{k}}{x_{k+m}}.$$

This computation along with Corollary 4.7 yields the desired result. □

Example 3.11. Let $n = 2$, and let $k = 0$ and $m = 2$. The $T$-Paths with respect to $M = M_{v_0,v_3}$ are illustrated as below.
Theorem 3.12. If $a \in CV$ then $x_M = \tilde{x}[a]$. Conversely every cluster variable is equal to $\tilde{x}[a]$ for some $a \in CV$.

Proof. It follows from Lemmas 3.8, 3.9 and 3.10. 

Definition 3.13. Two distinct elements $a, b \in CV$ are said to be disjoint if one of the following holds

1. $a_i = -1$ for some $i \in [1, n]$ and $b_i = 0$;
2. $b_i = -1$ for some $i \in [1, n]$ and $a_i = 0$;
3. $a_i = \begin{cases} 1, & \text{if } i \in [k + 1, k + m]; \\ 0, & \text{otherwise} \end{cases}$
   
   and

   and

   (a) $k \leq j$ and $j + l \leq k + m$ or
   (b) $j + l + 1 \leq k$ or
   (c) $k + m + 1 \leq j$

Corollary 3.14. Two distinct elements $a, b \in CV$ are disjoint if and only if $\tilde{x}[a]$ and $\tilde{x}[b]$ are in the same cluster.
Proof. Let $M$ and $N$ the two diagonals in $P$ that correspond to $\tilde{x}[a]$ and $\tilde{x}[b]$ respectively. In definition 4.13, in (1) $\tilde{x}[a] = x_i$, in (2) $\tilde{x}[b] = x_i$ and in both cases $a$ and $b$ are disjoint if and only if $N$ does not cross $M$ which is identical to saying that $\tilde{x}[a]$ and $\tilde{x}[b]$ are in the same cluster. In (3) none of $\tilde{x}[a]$ and $\tilde{x}[b]$ is equal to any of the $x_i$’s, and (a), (b) and (c) represent all the cases where $v_j$ and $v_{j+t+1}$ are both on the same side of $M$, i.e. they represent the cases where $M$ and $N$ do not cross which is equivalent to saying that $\tilde{x}[a]$ and $\tilde{x}[b]$ are in the same cluster. \qed

4. Cluster Monomials

Lemma 4.1. For any $a \in \mathbb{Z}^n$, $a$ can be written uniquely as

$$a = \sum_{j=1}^{m} k_j b_j$$

where $m \leq n$, $k_j \in \mathbb{Z}_{\geq 0}$, and the $b_j \in CV$ are disjoint.

Proof. For $a = (a_1, ..., a_n)$, let

$$L(a) = \{ (s, t) \in \mathbb{Z}^2 \mid 1 \leq s \leq n, \min(0, a_s) \leq t \leq \max(0, a_s) \}.$$ 

Each lattice point $(s, t)$ with $t < 0$ naturally gives a vector $(a_1, ..., a_n) \in CV$ with $a_s = -1$ and $a_i = 0$ for $i \in [1, n] \setminus \{s\}$. For each $t > 0$, let $L(a)_t = \{ s \in [1, n] \mid (s, t) \in L(a) \}$. Then each maximal connected interval of $L(a)_t$, say $[i, i+p]$, naturally gives a vector $(a_1, ..., a_n) \in CV$ with $a_1 = ... = a_{i-1} = 0, a_i = ... = a_{i+p} = 1, a_{i+p+1} = ... = a_n = 0$.

It is clear by construction that these vectors in $CV$, which we call $b_j$, are disjoint. \qed

Lemma 4.2. For any sequence $(b_1, ..., b_m)$ with disjoint $b_j \in CV$ and for any sequence $(k_1, ..., k_m)$ of positive integers, we have

$$\tilde{x}[a] = \prod_{j=1}^{m} \tilde{x}[b_j] = \prod_{j=1}^{m} (\tilde{x}[b_j])^{k_j}.$$ 

Proof. First we decompose $(D^{(i)})_{i \in [0,n]}$ as follows.

For $(s, t), (s+1, t) \in \mathbb{Z}^2$, let $\ell_{(s,t)}$ denote the line segment joining $(s, t)$ and $(s+1, t)$.

If $(s, t), (s+1, t) \in L(a)$ with $t < 0$ then assign to $\ell_{(s,t)}$ the empty path. Suppose $t < 0$. If $(s, t), (s+1, t) \in L(a)$ then assign to $\ell_{(s,t)}$ the subpath consisting of $u_t^{(s)}$ and $v_t^{(s)}$. If $(s, t) \in L(a)$ but $(s+1, t) \notin L(a)$ then assign to $\ell_{(s,t)}$ the edge $u_t^{(s)}$. If $(s+1, t) \in L(a)$ but $(s, t) \notin L(a)$ then assign to $\ell_{(s,t)}$ the edge $v_t^{(s)}$.

For each maximal connected interval, say $[i, i+p]$, of $L(a)_t$ with $t > 0$, let $D([i, i+p])$ be the collection of Dyck paths assigned to $\cup_{s=i}^{s=i+p} \ell_{(s,t)}$.

Then each GCC on $(D^{(i)})_{i \in [0,n]}$ can be decomposed into GCCs on the empty paths and (copies of) $D([i, i+p])$. Conversely GCCs on the empty paths and (copies of) $D([i, i+p])$
constitute a GCC on \((\mathcal{D}^{(i)})_{i \in [0,n]}\), because local compatibility is determined by subsets of corners. So the desired statement is obtained.

The main result follows from Theorem 3.12, Corollary 3.14, Lemma 4.1, and Lemma 4.2.

**Example 4.3.** Let \(n = 7\) and \(a = (1, 4, 4, 3, -2, 3, 3)\). Then the corresponding collection of Dyck paths is given as below

![Dyck path diagram]

We decompose these Dyck paths into subpaths

The collection of Dyck paths in each red box corresponds to a cluster variable. The above decomposition explains the following identity

\[
\tilde{x}[(1, 4, 4, 3, -2, 3, 3)] = \tilde{x}[(1, 1, 1, 0, 0, 0)](\tilde{x}[(0, 1, 1, 0, 0, 0)])^2(\tilde{x}[(0, 1, 1, 0, 0, 0, 0)])(\tilde{x}[(0, 0, 0, 0, 1, 1)])^3(\tilde{x}[(0, 0, 0, -1, 0, 0)])^2.
\]
5. Extending Definition 2.1 to a more general case

In an attempt to generalize our method as much as possible, in this section we show the equivalence of our definition of locally compatible pairs (definition 2.1), to a special case of the more general definition given in [5].

First, we define some necessary terminology as stated in [5]:

Let $D = D^{a_1 \times a_2}$. Let $D_1 = \{u_1, \ldots , u_{a_1}\}$ be the set of horizontal edges of $D$ indexed from left to right, and $D_2 = \{v_1, \ldots , v_{a_2}\}$ be the set of vertical edges of $D$ indexed from bottom to top. *Note that this indexing is different from the indexing given in Section 3. Given any points $A$ and $B$ on $D$, let $AB$ be the subpath starting from $A$, and going in the Northeast direction until it reaches $B$ (if we reach $(a_1, a_2)$ first, we continue from $(0,0)$). By convention, if $A = B$, then $AA$ is the subpath that starts from $A$, then passes $(a_1, a_2)$ and ends at $A$. If we represent a subpath of $D$ by its set of edges, then for $A = (i, j)$ and $B = (i', j')$, we have

$$AB = \begin{cases} \{u_k, v_l : i < k \leq i', j < l \leq j'\}, & \text{if } B \text{ is to the Northeast of } A; \\ D - \{u_k, v_l : i' < k \leq i, j' < l \leq j\}, & \text{otherwise}. \end{cases}$$

We denote by $(AB)_1$ the set of horizontal edges in $AB$, and by $(AB)_2$ the set of vertical edges in $(AB)$. Also, let $(AB)^o$ denote the set of lattice points on the subpath $AB$ excluding the endpoints $A$ and $B$ (here $(0,0)$ and $(a_1, a_2)$ are regarded as the same point).

Further, let $r$ be the number of edges between two vertices of a rank 2 equioriented quiver.

**Definition 5.1.** [5, Definition 1.10] For $S_1 \subseteq D_1$, $S_2 \subseteq D_2$, we say that the pair $(S_1, S_2)$ is locally compatible if for every $u \in S_1$ and $v \in S_2$, denoting by $E$ the left endpoint of $u$ and $F$ the upper endpoint of $v$, there exists a lattice point $A \in EF^o$ such that

$$|(AF)_1| = r|(AF)_2 \cap S_2| \text{ or } |(EA)_2| = r|(EA)_1 \cap S_1|.$$  

To motivate the proposition that these definitions are in fact equivalent for $r = 1$, consider $D^{2 \times 2}$, the maximal Dyck path from $(0,0)$ to $(2,2)$. Computing the locally compatible pairs of $D^{2 \times 2}$ using both definitions 5.1 and 2.1, we have $(\{u_1\}, \{v_2\})$ and $(\{u_2\}, \{v_1\})$ as the only non-trivial locally compatible pairs.

**Proposition 5.2.** The definitions 5.1 and 2.1 for local compatibility are logically equivalent for $r = 1$.

**Proof.** For the pair $(S_1, S_2) \subseteq D_1 \times D_2$, assume that for all $u_i \in S_1$ and all $v_j \in S_2$, there exists a lattice point $A$ such that either $|(AF)_1| = |(AF)_2 \cap S_2|$ or $|(EA)_2| = |(EA)_1 \cap S_1|$ holds, that is, the pair $(S_1, S_2)$ is locally compatible by definition 5.1. This directly implies that no horizontal edge $u_i \in S_1$ is the immediate predecessor of any vertical edge $v_j \in S_2$. Otherwise, if we have such a $u_i \in S_1$ and $v_j \in S_2$, then there will be only one lattice point $A \in EF^o$ arising from $u_i$ and $v_j$, which in turn will yield: $|(AF)_1| = 0$ and $|(AF)_2 \cap S_2| = 1,$
and also: $|(EA)_2|=0$ and $|(EA)_1 \cap S_1|=1$, which shows that such a pair does not satisfy our initial assumption. Thus, the pair $(S_1,S_2)$ is locally compatible by Definition 2.1.

Now we assume that for the pair $(S_1,S_2) \subseteq D_1 \times D_2$, no horizontal edge $u_i \in S_1$ is the immediate predecessor of any vertical edge $v_j \in S_2$, that is, the pair $(S_1,S_2)$ is locally compatible by Definition 2.1. Let $m$ be the slope of the main diagonal of the maximal Dyck path $D$. We proceed by cases:

**Case 1:** $m \leq 1$. Fix a $v_j \in S_2$, and let $F$ be the upper end point of $v_j$. The immediate predecessor of this $v_j$ will be a horizontal edge, by the definition of a maximal Dyck path. Let $A$ be the left end point of this horizontal edge (notice by our assumption this horizontal edge is $\notin S_1$). Since any $u_i \in S_1$ (and its left endpoint $E$) is before this horizontal edge (by our above definition of $EF$), we have $|(AF)_1|=1 = |(AF)_2 \cap S_2|$. 

**Case 2:** $m > 1$. Fix a $u_i \in S_1$. In this case, the immediate successor of $u_i$ will be a vertical edge. Let $A$ be the top end point of the vertical edge. Since any $v_j \in S_2$ is after this vertical edge, we have $|(EA)_2|=1 = |(EA)_1 \cap S_1|$. Thus, the pair $(S_1,S_2)$ is locally compatible by Definition 5.1. □

**References**


Department of Mathematics, Wayne State University, Detroit, MI 48202