First Take Home Exam

This is the first take home exam. You are allow to talk to anyone in the class, you are allowed to use your book and your notes. You cannot use any other book or talk to any person outside the class. At the end of your paper you need to clearly state whom you worked with. Also, I will be very careful on your presentations: writing has to be clear and efficient.

1. **A generalization of Cayley Theorem.**
   Let G be a group and H be a subgroup of G. Let $S = \{Hx \mid x \in G\}$ be the set of right cosets in G and let $A(S)$ be the group of permutations of S. Define $\lambda : G \to A(S)$ such that $\lambda(g)(Hx) = Hxg$. Prove the following:
   (a) (10 points) Prove that $\lambda(g) \in A(S)$ for all $g \in G$.
   (b) (10 points) Prove that λ is a group homomorphism.
   (c) (10 points) Assume that H is a normal group, show that $\ker(\lambda) = H$.
   (d) (10 points) Let $K = \ker(\lambda)$. Show that K is a normal subgroup of G and $K \subset H$.
   (e) (10 points) Let L be a normal subgroup of G such that $L \subset H$, show that $L \subset K$.
   (f) (10 points) Assume that G has finite order and that λ is an injective map, set $(G : H) = m$. Show that $|G|$ divides the $m!$. Conclude that if $|G|$ does not divide $m!$, then G must have a normal subgroup.

   This proves that the kernel of such a λ is the biggest normal subgroup contained in H. You will be surprised how powerful is this theorem.

2. (10 points) Let G be a group of order p^2. Use (1f) to show that there is a normal subgroup of order p in G.

3. **Every group of order p^2 is abelian: almost a proof**
   Prove the following steps:
   (a) (10 points) Let G be a group. Define $Z(G) := \{x \in G \mid xa = ax \quad \text{for every} \quad a \in G\}$. Prove that $Z(G)$ is a normal subgroup of G. $Z(G)$ is called the *center* of G.
   (b) (10 points) Let G be a group and a an element of G. Define $C(a) := \{x \in G \mid ax = xa\}$. Prove that $C(a)$ is a subgroup of G. $C(a)$ is called the *centralizer of* a.
   (c) (10 points) Let G be a group and $a \in G$ be an element. Show that $C(a) = G$ if and only if $a \in Z(G)$.
   (d) (20 points) Let G be a group such that $|G| = p^2$, where p is prime. Assume that $Z(G) \neq e$. Prove that G is abelian.

   In class we will prove that $Z(G)$ is always bigger than the trivial group $\{e\}$ if the order of G is p^2, where p is a prime. This will give us a proof that all groups of order p^2 are abelian. How many groups can you find of order 9? How many groups can you find of order 49? Make a guess: list all the possible groups of order p^2 for a prime p.

4. (20 points) Find a counterexample to the following statement: Let G be a group. If m is an integer such that m divides $|G|$, then there exists a subgroup of G such that $|H| = m$.

5. (10 points) Let G be a group without proper subgroups, show that G is cyclic and that the order of G is given by a prime number.