**Homework 2**

This is the list of exercises you can choose from for the second homework assignment. You need to score 30 points. Note I am not grading more than 30 points. Good luck and good fun.

1. (5 points) Let $(G, *)$ be a group and $H \subset G$ be a subset. Show that $H < G$ if and only if for every $a \in H$ and $b \in H$ one has $a * b^{-1} \in H$.
2. (5 points) Let $\phi : G \to H$ be a group homomorphism. Show that $\phi$ is 1-to-1 if and only if $\ker \phi = \{e_g\}$.
3. (5 points) Let $G$ be a cyclic group generated by an element $a$ (i.e. $G =< a >$). Show that $o(a) = |G|$.
4. (5 points) Let $G$ be a group and $a \in G$ be an element of finite order. Show that $o(a) = o(a^{-1})$.
5. (5 points) Let $G$ be a group of finite order. Show that $o(a)$ divides $|G|$ for all elements $a \in G$.
6. (5 points) Consider the group $(\mathbb{Z}_m, +)$. Show that the order of $[x] \in \mathbb{Z}_m$ is $m$ if and only if $x$ and $m$ are coprime.
7. (5 points) Let $(G, *)$ be a group. Prove that the map $i$ that sends $x$ to $x^{-1}$ is an automorphism if an only if the group is abelian.
8. (5 points) Write the following permutations as product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 3 & 7 & 10 & 9 & 6 & 1 & 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 3 & 6 & 1 & 4 & 2 & 10 & 7 & 8 & 9 \end{pmatrix}$$

9. Write as product of disjoint cycles, the product of the following cycles:

$$(1 \quad 2 \quad 3 \quad 4 \quad 7)(8 \quad 3 \quad 1 \quad 9)(2 \quad 8 \quad 7 \quad 1 \quad 3)$$

10. (10 points) Answer the following questions (with proof)
    (a) What is the order of a cycle of length $n$?
    (b) What is the order of the product of $k$ disjoint cycles of length $m_1, m_2, \ldots, m_k$?
    (c) How do you determine the order of a permutation?
11. (10 points) Show that there is only one group of order 5.
12. (10 points) Show that the $S_n$ is generated by $(1 \quad 2)$ and $(1 \quad 2 \quad 3 \quad \ldots \quad n)$.
13. (5 points) (Rachel) Show that $(\mathbb{R}^*, \dot{})$ is not cyclic.
14. (5 points) Show that the group of permutations $S_n$ has $n!$ elements.
15. (5 points) Let $(G, *)$ be a group. Show that the intersection of two subgroups of $G$ is a subgroup: if $H_1 \leq G$ and $H_2 \leq G$ then $H_1 \cap H_2 \leq G$.
16. (5 points) Let $\{A_i\}_{i \in I}$ be a partition of the set $A$. Show that there is an equivalence relation such that the equivalence classes are exactly the sets $A_i$.
17. (10 points) Let $(G, *)$ be a group. Let $a$ and $b$ two elements of the group $G$ such that $a^5 = e$ and $a * b * a^{-1} = b^2$. Find the order of $b$.