**Homework 1**

This is the list of exercises you can choose from for the first homework assigment. You need to score 30 points. Note I am not grading more than 30 points. Good luck and good fun.

1. (10 points) Show that a set with $n$ elements has $2^n$ different subsets.
2. (7 points) Let $A$ and $B$ two sets. Define $A + B := (A - B) \cup (B - A)$ and $A * B := A \cap B$. Show that
   (a) $A + B = B + A$.
   (b) $A + \emptyset = A$.
   (c) $A * A = A$.
   (d) $A + A = \emptyset$.
   (e) $A + (B + C) = (A + B) + C$.
   (f) If $A + B = A + C$ then $B = C$.
   (g) $A * (B + C) = A * B + A * C$.
3. (10 points) Prove that the set of integers is infinite [1].
4. (10 points) Assume that $S$ is an infinite set and that there exists a bijiection between $S$ and the set of integers $\mathbb{Z}$. Show that there exists a bijiection between $S$ and $S \times S$.
5. (7 points) Let $g, h : S \to T$ two functions and let $f : T \to U$ be an $1 - 1$ function. Show that if $f \circ g = f \circ h$ then $g = h$.
6. (10 points) Let $f : S \to S$ a $1 - 1$ function where $S$ is a finite set. Show that
   (a) $f$ is onto.
   (b) There exists an integer $n$ such that $f^n = i_S$, where $f^n$ is the composition of $f$ with itself, $n$ times ($f \circ \cdots \circ f$, $n$-times).
7. (30 points) Show that a group with 4 elements is isomorphic to either $(\mathbb{Z}_4, +)$ or $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, +)$.
8. (10 points)Let $G$ be a set of elements and $*$ a binary operation defined in $G$. We say that $(G, *)$ is a *left group* if the following hold:
   (a) $*$ is associative;
   (b) there exists an element $e \in G$ such that $e * x = x$ for every $x \in G$;
   (c) for every element $x \in G$ there exists an element $y \in G$ such that $y * x = e$.

   Prove that $(G, *)$ is a left group if and only if it is a group.
9. (5 points) Let $(G, *)$ be a group. Prove that for every element $x \in G$ there exists a unique element $y \in G$ such that $x * y = y * x = e$. (Uniqueness of the inverse)
10. (5 points) Let $\phi : G \to G'$ be a group homomorphism. Prove that $\phi(x^{-1}) = \phi(x)^{-1}$, for every $x \in G$.
11. (5 points) Let $(G, *)$ be a group. Prove that the set $H_G = \{a \in G \mid a*x = x*a \text{for all} \quad x \in G\}$ is a subgroup of $G$. (*The subgroup* $H_G$ *is called the center of* $G$)
12. (10 points) Find the number of elements of the following subgroups:

[1] Recall that in class we defined a set $S$ to be infinite if there is a bijiection between $S$ and a proper subset of $S$

(a) The subgroup of $(\mathbb{Z}_{30}, +)$ generated by the element [25].

(b) The subgroup of $(\mathbb{Z}_{25}, +)$ generated bu the element [10].

(c) The subgroup of $S_3$ generated by the permutation $(1 \quad 3)$.

(d) The subgroup of $(\mathbb{Z}_{63}, +)$ containing [21] and [3].

(e) The subgroup of $(M_{2\times 2}(\mathbb{Z}_6), +)$, generated by

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Let $H$ be a subgroup of $G$. Do you have any conjecture for the relation between the order of $H$ and the order of $G$?

13. (10 points) Let $r$ and $s$ two positive integers, Show that the set $H = \{sm + rn \mid n, m \in \mathbb{Z}\}$ is a cyclic subgroup of $\mathbb{Z}$. Find the generator.

14. (5 points) Which of the following are subgroups of $(GL_n(\mathbb{R}, \circ)$:

    (a) the $n \times n$ matrices with determinant equal to 1;

    (b) the $n \times n$ matrices with determinant equal to $-1$;

    (c) the $n \times n$ matrices with determinant equal to 2;

    (d) the set of $n \times n$ orthogonal matrices;

    (e) the set of $n \times n$ diagonal invertible matrices.

15. (20 points) Let $G$ be a finite group and let $\phi : G \to G$ be an automorphism such that $\phi(x0 = x$ if and only if $x = e$. Assume that $\phi \circ \phi = id$. Show that $G$ is abelian.

16. (5 points) Decide if the following are group homomorphisms:

    (a) $\phi : (\mathbb{R}, +) \to (\mathbb{R}, +)$, such that $\phi(x) = x + 3$, for all $x \in \mathbb{R}$;

    (b) $\phi : (\mathbb{R}, +) \to (\mathbb{R}, +)$, such that $\phi(x) = 3x$, for all $x \in \mathbb{R}$;

    (c) $\phi : (\mathbb{R}, +) \to (\mathbb{R}, +)$, such that $\phi(x) = x^2$, for all $x \in \mathbb{R}$;

    (d) $\phi : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{R} \setminus \{0\}, \cdot)$, such that $\phi(x) = x^2$;

    (e) $\phi : (\mathbb{C}, +) \to (\mathbb{R}, +)$, such that $\phi(x + iy) = x$, for all $x \in \mathbb{R}$;

    (f) $\phi : (\mathbb{C}, +) \to (\mathbb{R}, +)$, such that $\phi(x + iy) = x$, for all $x + iy \in \mathbb{C}$;

    (g) Let $V$ ve a vector space and $c$ a scalar. $\phi : (V, +) \to (V, +)$, such that $\phi(\mathbf{v}) = c\mathbf{v}$ for all $\mathbf{v} \in V$.

17. (10 points) Let $G$ be a group and $g \in G$. Let $\phi : G \to G$ the map defined by sending $x$ to $g * x * g^{-1}$. Show that $\phi$ is a one-to-on and onto homomorphism.