M314 December 16, 2005 Name:________________________________

Instructions: You may use a calculator but no book or notes.

[1](15 points) Prove that $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}-\mathbf{v}||$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

[2](15 points) Let $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (0, 1, 1)$.

(a) Write down and solve a system of equations for all vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(b) Find a linear equation for the plane $\mathcal{P}$ through the three points $P = (2, 2, 2)$, $Q = (3, 3, 2)$ and $R = (2, 3, 3)$. (Note that the vector from P to Q is $\mathbf{u}$ and the vector from P to R is $\mathbf{v}$.)

(c) Find the point on $\mathcal{P}$ closest to the origin, $(0, 0, 0)$.

[3](15 points) Let $A = \begin{pmatrix} 2 & 0 & 6 \\ 2 & 4 & -2 \\ 0 & -b & 2 \end{pmatrix}$.

(a) For what value or values of b is A the augmented matrix of an inconsistent linear system of equations? Explain how you know A is inconsistent for the values of b that you give and for no others.

(b) For what value or values of b is A invertible? Explain how you know.

[4](15 points) Let $A = \begin{pmatrix} 2 & 0 & 6 \\ 2 & 4 & -2 \\ 0 & -b & 2 \end{pmatrix}$.

(a) For what value or values of b is A the augmented matrix of an inconsistent linear system of equations? Explain how you know A is inconsistent for the values of b that you give and for no others.

(b) For what value or values of b is A invertible? Explain how you know.

[5](15 points) Consider $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 3 & 6 & 2 & 2 & 3 & 2 \end{pmatrix}$ whose reduced form is $R = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find a basis of the nullspace of A.

(b) Find a basis of the column space of A.

(c) Find a basis of the row space of A.

[6](15 points) Give an example of each of the following:

(a) A matrix that is not diagonalizable. Explain how you know the matrix is not diagonalizable.

(b) A matrix that is diagonalizable but not diagonal. Explain how you know the matrix is diagonalizable.

(c) A 3×4 matrix of rank 2.

(d) A 3×4 matrix of nullity 1.

(e) An orthogonal matrix.

[7](15 points) Let $V = \text{Span}([1,-1,0],[1,0,-1]) \subset \mathbf{R}^3$.

(1) Find an orthogonal basis for V.

(b) Find an orthogonally diagonalizable matrix A such that $A[1,1,1]^T = [3,3,3]$ and such that the span of the the eigenvectors with eigenvalue 0 is V. (You may write your answer as PDP^T if you explicitly find an appropriate P and D.)

[8](15 points) For each of the following statements, indicate whether it is TRUE or FALSE. If it is true, explain why; if it is false, give a counterexample.

(a) If A is an $n \times n$ matrix such that $A^2 = A$, then A is either the zero matrix or the identity matrix.

(b) If A is an $n \times n$ matrix of rank n such that $A^2 = A$, then $A = I_n$.

(c) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^3$, then $||\mathbf{v}+\mathbf{w}|| \geq ||\mathbf{v}|| + ||\mathbf{w}||$.

(d) If A is a square matrix, then it is never true that $\det(-A) = \det(A)$.

(e) If A is a square matrix, then it is always true that $\det(2A) = 2\det(A)$.

[9](15 points) Recall that $\mathcal{P}_2 = \text{Span}(1, x, x^2)$ is a real vector space.

(a) Let S_0 be the set of all polynomials $f(x)$ in $\mathcal{P}_2 = \text{Span}(1, x, x^2)$ such that $f(1) = 0$. Is S_0 a subspace of $\mathcal{P}_2$? Explain why or why not.

(b) Let S_1 be the set of all polynomials $f(x)$ in $\mathcal{P}_2 = \text{Span}(1, x, x^2)$ such that $f(0) = 1$. Is S_1 a subspace of $\mathcal{P}_2$? Explain why or why not.