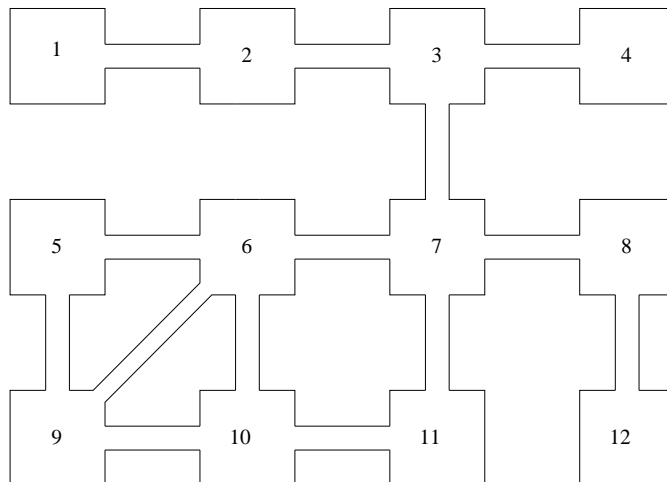


Robot in a Maze

Class Project — Math 314–003 — Fall 2006

Due: December 1, 2006

A robot is placed in the maze below and is programmed to move at random. Each minute, the robot either stays in the room it is currently located or chooses a hallway at random and moves into the adjoining room, and each possibility is equally likely. For example, if the robot happens to be in room 11, then there is a $\frac{1}{3}$ chance it will move next to room 10, a $\frac{1}{3}$ chance it will move next to room 7, and a $\frac{1}{3}$ chance it will stay in room 11.



1. Study the short- and long-term behavior of the robot in this maze. Specifically, consider the following topics in detail.
 - Find the appropriate transition matrix P describing the movement of the robot.
 - If \vec{v} is a column vector whose i -th entry gives the likelihood that the robot is currently in room i , what does $P\vec{v}$ tell you and why?
 - Carefully explain the real-world significance of the matrix P^2 . Be sure to justify, in your own words, the meaning of the entries of the matrix P^2 . To answer this question, you should think carefully about how matrix multiplication is defined. Do the same thing for P^3 , P^4 , etc.
 - An important concept in studying stochastic matrices is that of “regularity”. A stochastic matrix is called *regular* if some power of it has no zero entries. Show that P is regular. Also, find the smallest integer k such that P^k has no zero entries. What is the practical meaning (in terms of properties of the maze) of the value of k ?
 - Explain the significance of P^k for k a very large integer.
 - What is the approximate probability that the robot will wind up in each of the twelve rooms after a very long time has passed?
 - Does your answer to the previous part depend on where the robot begins his journey?
 - Find the exact steady-state vector for this system by solving an appropriate system of equations.
 - What connection is there between the entries of the steady-state vector and the nature of the maze? Can you explain why some of the entries of the steady-state vector are equal to others?
2. Suppose the maze is now modified so that the hallway joining rooms 3 and 7 is permanently blocked off. This leads to a new transition matrix — call it Q .
 - Is Q regular? How can you be sure of your answer?
 - What is the “real-world” interpretation of your answer to the previous question?
 - Does the system have one or more steady-state vectors in this situation? If so give the steady-state vector or vectors. Interpret your answers here in “real-world” terms.
 - What if now the hallway joining rooms 6 and 7 and the hallway joining rooms 10 and 11 are also blocked (and the hallway joining rooms 3 and 7 remains blocked). Find all the steady-state vectors for this new situation and explain their significance in terms of the original matrix.
 - What is the general pattern here — how many steady-state vectors will there be for a given maze of this sort?

3. Now return to the original maze (with rooms 3 and 7, 6 and 7, and 10 and 11 all still connected). Suppose a leak has formed in the ceiling of room 12, and each time the robot enters that room, there is a one percent chance that water will drip onto the robot's circuitry and ruin it. Find a way to model this mathematically (there are several ways) and address the following questions:
- What is the probability that the robot will still be functioning after, say, 50 minutes have passed for each of the 12 possible starting rooms?
 - What can you say about the long term fate of the robot?
4. Return now to the situation in part (2), in which the hallway joining rooms 3 and 7 is blocked off. Assume the robot starts in one of the rooms 1 through 4 (and so you can ignore rooms 5 through 12). But now modify the programming of the robot as follows: the robot is no longer allowed to stay put but rather chooses always to move at random down one of the hallways available to it.
- Find the 4×4 stochastic matrix for this situation.
 - Is this matrix regular? Why or why not?
 - Does this matrix have a steady-state vector?
 - How do your answers to these questions relate to the nature of the maze and the robot's programming?

consider the 4×4 stochastic matrix describing the movement of the

Computers: You can conceivably do this project using just a fancy calculator. I encourage you, however, to use the software package *Maple* (or an equivalent one). For one thing, it is far easier to enter, manipulate, and view large amounts of data in *Maple* than it is on a calculator, and this project will require the use of rather large matrices. Also, I believe it is a valuable experience to learn how to use a computer algebra system such as *Maple*.

Maple is available on the Matlab computer system and everyone in class should have an account on Matlab. The Matlab is located in room 18 of Avery Hall and you can see what its hours are at the website:

<http://www.math.unl.edu/pi/studentResources/labs>.

Your login and password for the computers in the Matlab should be the same as for your usual university "active directory" account.

Here are some helpful tips concerning *Maple*. These are based on Version 10 of *Maple*.

- You will need to "import" the linear algebra package into *Maple*. Enter the command `with(LinearAlgebra):` at the very start of your *Maple* session. (The trailing colon suppresses a long list of commands that are imported with the *LinearAlgebra* package — to see this list, replace the colon with a semi-colon.)
- To learn more about a *Maple* command or topic, type in the command name or topic preceded by a question mark. For example, try entering `?Matrix` into *Maple*. (You can also use the menu-driven help documentation.)
- Here are a couple of useful commands:
 - The command `B := Matrix([[a,b,c],[d,e,f]]);` sets B to be the evident 2×3 matrix.
 - You can use plus, minus, and exponents in the usual manner. Instead of using an asterisk for multiplication, however, use the period key. For example, enter A and B to be your favorite 3×3 and 2×3 matrices, and try `B.(A+3A^2);`.
 - Suppose A is a matrix with rational numbers as entries and you would rather have them as decimals. Then you could use the command `map(evalf,A);`. Suppose you want to round off the entries of A to 3 decimal places of accuracy. (This is useful for printing a large matrix so that it fits on one page.) Then use `map(x -> evalf(x,3),A);`.
 - By default, *Maple* only displays in full matrices of size at most 10×10 . To increase this to 12, use `interface(rtablesize = 12)`.

Important Writing Instructions: After you have analyzed the mathematics of this maze problem, write a well-written report summarizing your findings. This report should be written to form a clear, well-organized description of what you have learned, accessible to someone who knows nothing about the specific project you have been assigned. In particular, do **not** treat the above itemized list as some sort of worksheet to be filled in with unconnected answers. (In fact, you do not necessarily have to address every issue in the above itemized list — I leave it up to your judgment.) Instead, strive to write a report that stands on its own and can be read and understood by someone who is not taking our class. To be more specific, *pretend the reader of your report is someone who has taken Math 314 in the distant past and who does not have a copy of this assignment.* In particular, *do not write this as if I am the only one reading it!*

One general tip is that it helps to be as explicit as possible. That is, in addition to stating general properties and observations, reinforce them with examples which are as explicit as possible. For example, when describing the meaning of P^2 , in addition to stating the general principle, refer to a specific entry for clarification.

Grading: This project is worth 100 points. Of these, 75 points will be for mathematical content. The remaining 25 points will be based on clarity of exposition. I *will* take off for things like poor grammar, spelling errors, awkward style, unclear writing, etc.

Deadline: The final deadline for this project is Friday, December 1. This is about six weeks away, but I would strongly encourage you not to put it off. In particular, if you give me a rough draft well before the due date, I will look it over and offer feedback.

Groups: Here are the groups we formed in class on October 20. If you are unhappy with these group assignments (e.g., if you were not in class that day) let me know as soon as possible.

Pat Yoakum Jacob Martin Kurt Larson	Chris Myers Kyle Gerking Brett Shoup	Jacob Millard Nick Tuck Sean Campbell
Lee Matejka Jessica Sierk Dan Herlacher	Scott Heuke Wes Meredith Matt Peterson	Kyle Scheele Jay Schaefer Andrew Shroll
Malinda Lammers Ashley Grace Tim Seberger	Tony Filips Aaron Wentzel Tyler Hehn	Mike Merwald Jon Soneson Nate Rasmussen
Bill Rice Stan Schnell John Schonert	Jay Carlson	