Math 314: Linear Algebra - Fall 06
Final: Linear Algebra Name ____________________

1. (22 points) In the folling A and B are $n \times n$ matrices. Mark True of False the following statements.

   (a) If A is not zero then $\det(A) \neq 0$.

   (b) If A is real and symmetric then its eigenvalues are real.

   (c) If $\det(AB) \neq 0$ then A is invertible.

   (d) If the reduced echelon form of $A - 5\,\mathrm{Id}$ is the $n \times n$ identity matrix then 5 is not an eigenvalue.

   (e) Let $\mathbf{b}$ a column vector of R^n. If the system $A\mathbf{x} = \mathbf{b}$ has no solution then $\det(A) \neq 0$.

   (f) Let C be a 3×5 matrix. The rank of C could be 4.

   (g) Let C be a $n \times m$ matrix, and $\mathbf{b}$ a column vector of R^n. If $C\mathbf{x} = \mathbf{b}$ has no solution then $\mathrm{rank}(C) < n$.

   (h) Let $W = \mathrm{span} < \mathbf{v}_1, \mathbf{v}_2 > \subset R^n$ be a vector space and $\mathbf{v}$ be a vector in R^n. Then $proj_W(\mathbf{v}) = proj_{\mathbf{v}_1}\mathbf{v} + proj_{\mathbf{v}_2}\mathbf{v}$.

   (i) Any diagonalizable matrix is invertible.

   (j) If A is invertible then it is similar to the identity.

   (k) If A is invertible then its reduce echelon form is the identity.

2. (21 points) Let W be the subspace of R^4 spanned by the vectors in $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Explain why $\mathcal{B}$ is an orthogonal basis for W.

(b) What is the dimension of W? Explain your answer.

(c) What is the dimension of $W^\perp$? Explain your answer.

(d) Find a basis for $W^\perp$.

(e) Is $\begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \end{bmatrix}$ in $W^\perp$?

(f) Is $\mathbf{v} = \begin{bmatrix} -5 \\ 7 \\ 2 \\ -5 \end{bmatrix}$ in W? If so, find $[\mathbf{v}]_{\mathcal{B}}$.

(g) What is $W \cap W^\perp$?

3. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of B are $\lambda = 0$ and $\lambda = 2$. Use these information to answer the following questions. For each matrix you list, give an explanation. There may be more than one matrix that satisfies the given condition.

(a) Which matrices are invertible?

(b) Which matrices have a repeated eigenvalue?

(c) Which matrices have rank less than 3?

(d) Which matrices are diagonalizable?

(e) Which matrices are orthogonally diagonalizable?

4. (33 points) Let A be a 4×7 matrix, such that the reduced echelon form is given by

$$R := \begin{bmatrix} 0 & 1 & 2 & 4 & 7 & -3 & 6 \\ 0 & 0 & 1 & 3 & 6 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A?

(b) What is the rank of R?

(c) Give a basis for $\text{row}(A)$.

(d) Give a basis for $\text{row}(R)$.

(e) Give a basis for $\text{col}(A)$.

(f) Give a basis for $\text{col}(R)$.

(g) Which columns of A are a basis for $\text{col}(A)$?

(h) What is the the nullity of A?

(i) Find a basis for $null(A)$.

(j) If $\mathbf{v}$ is the column vector $[1, 1, 1, 1]^T$, does $A\mathbf{x} = \mathbf{v}$ have a solution?

(k) Do the columns of A span R^4?

5. (20 points) Let A be a 3×3 matrix. Assume that the eigenvalues are 1 and 0 and the eigenspace E_1 is generated by $[1, 0, 1], [0, 0, 1]$ while E_0 is generated by $[1, 1, 2]$.

   (a) Is A diagonalizable? If yes, write the diagonal matrix D and the matrix P, such that $A = P^{-1}DP$.

   (b) Find A.

6. (20 points) Consider the linear trasformation T from R^n to R^n, such that $T(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{w}$, where $\mathbf{w} = [1, 1, \cdots, 1]]$.

   (a) Find the matrix A_T associated to the linear trasformation T.

   (b) Set $n = 10$, find the determinant of the matrix A_T.

7. (40 points) Let $\mathcal{P}_2$ be the vector space of the polynomials in one variable of degree at most 2.

(a) Explain why $\mathcal{B}_1 = \{1, 2 - x, 3 - x^2, x + 2x^2\}$ is not a basis for $\mathcal{P}_2$.

(b) Extract from $\mathcal{B}_1$ a basis for $\mathcal{P}_2$, call it $\mathcal{C}$.

(c) Let $\mathcal{B}$ be the basis given by $1, x, x^2$. Compute the matrix of change of basis from $\mathcal{B}$ to $\mathcal{C}$:$P_{\mathcal{C}\leftarrow\mathcal{B}}$.

(d) Consider the vector $\mathbf{v} = 1 + 3x + x^2$, give $[\mathbf{v}]_{\mathcal{C}}$.

8. (14 points) Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3. Let $\mathcal{A} = \{p(x) \in \mathcal{P}_3 \mid p(1) = p'(1) = 0\}$. Find a basis for $\mathcal{A}$.

9. (10 points) What topic covered in the course was the most difficult for you to understand? Why?

10. (10 points) What was your favorite part of this course? Why?