Review Problems for Exam 2

This is a list of problems to help you review the material which will be covered in the second exam. Please go over the problem carefully. Keep in mind that I am going to put some problems that are part of what was covered in the first exam. It is a good idea to re-work the problems from the review sheet for the first exam. Enjoy, and have a wonderful fall break.

(1) (a) Show that the two functions

$$f(x) = 2\sin^2 5x \quad g(x) = 3\cos^2 5x$$

are linearly independent on the real line.
Solution *If* $A2\sin^2 5x + B3\cos^2 5x = 0$ *for* $x = 0$ *we obtain* $2A = 0$ *which implies that* $A = 0$. *For* $x = \pi/2$ *we obtain* $3B = 0$ *which implies that* $B = 0$.

(b) Find a non-trivial linear combination of the functions

$$f(x) = 2\sin^2 5x \quad g(x) = 3\cos^2 5x \quad h(x) = 6$$

that vanishes identically on the real line. Does this contradict part (a)?
Solution $\frac{18}{2}(2\sin^2 5x) + 6(3\cos^2 5x) - 1(6) = 0$. *We did not contradict the statement in (a) since there is a new function.*

(2) Solve the initial value problem

$$y'' - 2y' + 2y = 2x; \quad y(0) = 1, \; y'(0) = 5.$$

Solution *The solutions of the characteristic polynomial* $r^2 - 2r + 2 = 0$ *are given by* $r = 1 \pm i$. *So the general solution of the homogeneous equation is given by*

$$y(x) = e^x(A\cos x + b\sin x)$$

A particular solution of the differetial equation is given by the linear combination $f(x) = c_1 + c_2 x$, *whose first derivative is* $f'(x) = c_2$ *and second derivative* $f''(x) = 0$. *In particular we obtain*

$$-2(c_2) + 2(c_1 + c_2 x) = 2x,$$

which yields $c_2 = 1$ *and* $c_1 = 1$. *The general solution for the differential equation is*

$$y(x) = e^x(A\cos x + b\sin x) + 1 + x.$$

For $x = 0$ *we obtain* $1 = A + 1$ *which gives* $a = 0$. *The first derivative of* $e^x(b\sin x) + 1 + x$. *is given by* $y'(x) = e^x(b\sin x + b\cos x) + 1$. *For* $x = 0$

$$y'(0) = 5 = b + 1$$

which implies $b = 4$.
The solution for the initial value problem is $\boxed{y(x) = e^x(4\sin x) + 1 + x.}$

(3) Consider the differential equation

$$y^{(3)} + y'' = 0.$$

(a) Find the general solution.

Solution *The characteristic polynomial is given by* $r^3 + r^2 = 0$, *wo that the solutions are* $r = 0$ *counted two times and* $r = -1$. *The general solution of the differential equation is given by* $\boxed{y(x) = A + Bx + Ce^{-x}.}$

(b) Show that all solutions with $y(0) = 2$ have the form

$$A + Bx + (2 - A)e^{-x},$$

where A and B are real constants.

Solution *The initial condition* $y(0) = 2$ *forces* $2 = A + C$ *and therefore* $y(x) = A + Bx + (2 - A)e^{-x}$

(c) Find a solution with $y(0) = 2$, $y'(0) = 0$ and $y''(0) = 0$. **Solution** *The initial condition* $y'(0) = 0$ *forces* $B = 2 - A$. *The initial condition* $y''(0) = 0$ *forces* $2 - A = 0$. *Therefore* $A = 2$ *and* $B = 0$.

(4) Consider the non-homogeneous linear differential equation

$$3y' + 2y = 2. \qquad (**)$$

(a) Write down the corresponding homogeneous equation.

Solution $3y' + 2y = 0$

(b) Write down the characteristic equation to and solve it.

Solution $3r + 2 = 0$, *which yields* $r = 2/3$

(c) Find the general solution the homogeneous equation.

Solution $y(x) = Ae^{2/3x}$

(d) Find by inspection a particular solution to (∗∗).

Solution *The constant function* $y = 1$ *is a solution to* (∗∗).

(e) Write down the general solution to (∗∗).

Solution $y(x) = Ae^{2/3x} + 1$

(5) Find the general solution of the equation $9y^{(3)} + 11y'' + 4y' - 14y = 0$ knowing that $y = e^{-x} \sin x$ is a solution.
Solution *The characteristic polynomial is given by* $9r^3 + 11r^2 + 4r - 14 = 0$ *and since* $e^{-x} \sin x$ *is a solution of the differential equation, we know that* $-1 + i$ *is a solution of the characteristic polynomial.Because the characteristic polynomial has real coefficients we also know that* $1 - i$ *is a solution. This implies that the polynomial* $(r - (-1 + i))(r - (-1 - i)) = r^2 + 2r + 2$ *divides the characteristic polynomial. By the long division algorithm, the quotient is given by* $9r - 7$*. So the third solution of the characteristic polynomial is given by* $r = 7/9$ *and the general solution is*
$$\boxed{y(x) = e^{-x}(A \cos x + B \sin x) + Ce^{7/9x}}$$

(6) Find a particular solution of the differential equation $4y'' + 4y' + y = 3xe^x$. Then write the general solution of such equation.

(7) Find a particular solution of the differential equation $y'' + 3y' + 2y = 4e^x$. Then write the general solution of such equation.

(8) Write all the solutions of the equation $x^{10} + 1 = 0$.
Solution *If a complex number* x *is such that* $x^{10} = -1$*, then* $x = \cos \alpha + i \sin \alpha$ *such that*
$$\cos(10\alpha) + i \sin(10\alpha) = \cos(\pi) + i \sin(\pi),$$
so that $\alpha = \frac{\pi}{10} + \frac{2}{10} k\pi$ *for any integer value of* k*. From here one computes the all* 10 *solutions.*

(9) Show that the functions $f(x) = x^2, g(x) = x$ and $h(x) = 3x + 5x^2$ are linearly dependent.
Solution $-5(x^2) - 3(x) + 1(3x + 5x^2) = 0$*, so there is a linear combination equal to zero where the coefficients are not all zero.*

(10) Consider the differential equation $x^2y'' + xy' - 9y = 0$ with $y > 0$. Check that $y_1(x) = x^3$ is a solution of such equation. Then, use the method of reduction of order to find a second, linearly independent solution.

(11) Consider the system of first order differential equations associated to $x^{(4)} + 3x^{(3)} - x = \sin t$. Do not solve the system.

(12) Solve the system of linear differential equations $x' = 3y$, $y' = 3x + y$.