Review Problems for Exam 2

This is a list of problems to help you review the material which will be covered in the final. Go over the problem carefully. Keep in mind that I am going to put some problems that are part of what was covered in the first exam. It is a good idea to re-work the problems from the review sheet for the inclass exam. Enjoy, and have a wonderful fall break.

1. First-order equations
   (a) Separable, first-order linear equations (method of integrating factors). Exact equations.
   (b) Substitution methods, in particular $v = \frac{y}{x}$
   (c) Reducible second-order equations
   (d) Euler's method
2. Modeling
   (a) Natural growth (and radioactive decay)
   (b) Logistic population growth (including harvesting)
   (c) Newton's law of cooling
   (d) Acceleration-velocity models (including friction)
   (e) Mixture problems.
3. General linear equations
   (a) Homogeneous equations with constant coefficients
   (b) In homogeneous equations, undetermined coefficients, reduction of order, variation of parameters.
4. Systems
   (a) Representing systems using matrix notation
   (b) Reducing a system to a single equation
   (c) Eigenvectors and eigenvalues
   (d) Defective eigenvalues and generalized eigenvectors
5. Stability
   (a) Critical points, phase diagrams (and phase portraits)
   (b) Stable and unstable critical points, nodes
6. Laplace transform
   (a) The definition, and the linearity property.
   (b) The inverse of Laplace transform.
   (c) Translation rules

(d) Laplace transform of derivatives and applications to solutions of differential equations and systems.

Problems

The following is a list of problems that can help you prepare for the final exam. keep in mind that I am assuming that you worked out the problem from the homework. In particular I might include any problem from the homework set.

1. Suppose functions $y(t)$ and $z(t)$ satisfy $x'' + x = 0$. Show that $2y(t) + 3z(t)$ is also a solution.

2. Consider the differential equation

$$\frac{dy}{dx} = \sqrt{x - y}. \qquad (*)$$

   (a) Verify that the function

$$y(x) = x - 1$$

   is a solution to $(*)$ on the whole real line.

   (b) There is an existence and uniqueness theorem for solutions to first order differential equations. Does this theorem guarantee existence of a solution to $(*)$ with $y(2) = 2$? If so, is it unique?

3. Given $y' - 2y = 0$

   (a) Find constant solutions.

   (b) Find the general solution treating the equation as separable

   (c) Using integrating factors

   (d) As a general first-order linear equation

   (e) Using the Laplace transform

   (a) Which of the previous methods would not apply to $y' + e^x y = e^x$

   (b) What about

$$(x - 2)y' = \frac{x}{y(x+3)} \qquad ?$$

4. Find the solution of the differential equation $\frac{dy}{dx} e^{2x-y}$. Write down which technique you applied.

5. Find the solution of the linear first-order differential equation $(x^2 + 1)\frac{dy}{dx} + 3xy = x$. Solve the initial value problem with $y(0) = 1$.

6. Consider the autonomous differential equation

$$\frac{dx}{dt} = 13x - 36 - x^2. \qquad (*)$$

   (a) Find the critical points.

(b) For each of the critical points, determine whether it is stable or unstable; draw the phase diagram for $(*)$.

(c) Scketch the slope field of $(*)$.

7. Apply Euler's method to the initial value problem

$$y' = 2, \quad y(0) = 1$$

first with step-size 1, then with step 0.5 to compute $y(1)$. What if you used any other step size? Explain.

8. A cup of instant coffee on a kitchen table has temperature 190°. Periodically the coffee is stirred with a plastic spoon and reaches a temperature of 150° after 3 minutes. Approximately when was the boiling water poured into the cup?(*The answer should come out to be* -1.25)

9. Solve the initial value problem

$$\frac{dy}{dx} = 3x^2y + 2xe^{x^3}; \quad y(1) = 0.$$

Solution *Use the integrating factor method. First rewrite the equation as:*

$$\frac{dy}{dx} - 3x^2y = 2xe^{x^3}.$$

The integrating factor is

$$\rho(x) = e^{-\int 3x^2dx} = e^{-x^3}.$$

After multiplication with $\rho(x)$ *the equation reads*

$$e^{-x^3}\frac{dy}{dx} - e^{-x^3}3x^2y = 2x.$$

We recognize the left-hand side as $D_x(e^{-x^3}y)$. *The general solution therefore is*

$$y = e^{x^3}\int 2xdx = e^{x^3}(x^2 + C),$$

where C *is a real constant.*

Finally,

$$0 = y(1) = e(1 + C)$$

shows that the desired solution has $C = -1$, *that is*

$$y = e^{x^3}(x^2 - 1).$$

10. Use Laplace transforms to solve the initial value problem

$$x'' + 4x' + 3x = 0; \quad x(0) = -1 \text{ and } x'(0) = 0.$$

Solution *Take the Laplace transform of both sides of the equation:*

$$\begin{aligned} [s^2X(s) + s] + 4[sX(s) + 1] + 3X(s) &= 0 \\ (s^2 + 4s + 3)X(s) &= -s - 4. \end{aligned}$$

Thus,

$$X(s) = \frac{-s-4}{s^2+4s+3}.$$

Factoring the denominator and using partial fractions we get

$$\begin{aligned} X(s) &= \frac{-s-4}{(s+1)(s+3)} \\ &= \frac{-3/2}{s+1} + \frac{1/2}{s+3} \\ &= -\frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3}. \end{aligned}$$

Finally, take the inverse Laplace transform to get the desired solution

$$x(t) = -\tfrac{3}{2}e^{-t} + \tfrac{1}{2}e^{-3t}.$$

11. A runner in the Lincoln Marathon starts out too fast; as consequence her speed decreases throughout the race at a rate inversely proportional to the square root of time.

  (a) Set up the differential equation that describes how her speed depends on time.
  (b) The runner starts out running 8 mph, and after one hour her speed has reduced to 7 mph. Find her speed as a function of time.
  (c) Find the function that expresses her distance from the starting line as a function of time.

Solution *(a) Let $s = s(t)$ be the runner's position (distance from the starting line) at time t and $v = v(t) = \frac{ds}{dt}$ her speed. The differential equation is then*

$$\frac{dv}{dt} = \frac{-k}{\sqrt{t}}, \qquad (*)$$

where k is a positive constant.

(b) The given conditions translate to $v(0) = 8$ and $v(1) = 7$. We solve the equation $(*)$ by integrating each side with respect to t

$$v = \int \frac{-k}{\sqrt{t}}\,dt = -2k\sqrt{t} + C$$

and use the initial conditions to find the constants k and C:

$$8 = v(0) = -2k\sqrt{0} + C = C$$

so $C = 8$ and therefore

$$\begin{aligned} v &= -2k\sqrt{t} + 8 \\ 7 &= v(1) = -2k\sqrt{1} + 8 = -2k + 8 \\ 2k &= 1 \\ k &= \frac{1}{2} \end{aligned}$$

Thus,

$$v(t) = -\sqrt{t} + 8.$$

(c) We integrate once more and use the initial condition $s(0) = 0$:

$$\begin{aligned} s &= \int (-\sqrt{t} + 8)\, dt = -\frac{2}{3}t^{3/2} + 8t + C_1 \\ 0 &= s(0) = C_1 \end{aligned}$$

Thus,

$$s(t) = -\frac{2}{3}t^{3/2} + 8t.$$

12. Consider the following system of differential equations:

$$\frac{dx_1}{dt} = 2x_1 + 4x_2 - 12e^{-2t} \quad \text{and} \quad \frac{dx_2}{dt} = x_1 + 2x_2 \qquad (**)$$

(a) Write the system $(**)$ in matrix notation.

(c) Find the general solution to $(**)$.

(b) Find the solution to $(**)$ with $x_1(0) = 2$ and $x_2(0) = 2$.

Solution *(a) In matrix notation $(**)$ reads*

$$\mathbf{x}' = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \mathbf{x} \quad - \quad \begin{bmatrix} 12e^{-2t} \\ 0 \end{bmatrix}.$$

(b) First we solve the homogeneous equation

$$\mathbf{x}' = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \mathbf{x}. \qquad (*)$$

The characteristic polynomial

$$(2 - \lambda)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

has roots 0 and 4. These are the eigenvalues of the coefficient matrix. An eigenvector associated to 0 is

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

and an eigenvector associated to 4 is

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus, the general solution to $(*)$ *is*

$$\mathbf{x}_c = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 e^{4t} = \begin{bmatrix} -2c_1 + 2c_2e^{4t} \\ c_1 + c_2e^{4t} \end{bmatrix},$$

where c_1 *and* c_2 *are real constants.*

Since there is no duplication with the complementary function $\mathbf{x}_c$*, we can find a particular solution to* $(**)$ *by determining the vector* $\mathbf{a}$ *in the trial solution*

$$\mathbf{x}_p = \mathbf{a}e^{-2t} = \begin{bmatrix} a_1e^{-2t} \\ a_2e^{-2t} \end{bmatrix}.$$

Since

$$\mathbf{x}_p' = \begin{bmatrix} -2a_1e^{-2t} \\ -2a_2e^{-2t} \end{bmatrix}$$

the equation

$$\mathbf{x}_p' = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\mathbf{x}_p \quad - \quad \begin{bmatrix} 12e^{-2t} \\ 0 \end{bmatrix}$$

reads

$$\begin{bmatrix} -2a_1e^{-2t} \\ -2a_2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2a_1e^{-2t} + 4a_2e^{-2t} - 12e^{-2t} \\ a_1e^{-2t} + 2a_2e^{-2t} \end{bmatrix}.$$

The last row yields

$$-2a_2 = a_1 + 2a_2 \quad \text{so} \quad a_1 = -4a_2,$$

and the first one then gives

$$-2a_1 = 2a_1 + 4a_2 - 12 = a_1 - 12 \quad \text{so} \quad a_1 = 4 \;\; \text{and} \;\; a_2 = -1.$$

The general solution to $(**)$ *is then*

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = \begin{bmatrix} -2c_1 + 2c_2e^{4t} + 4e^{-2t} \\ c_1 + c_2e^{4t} - e^{-2t} \end{bmatrix},$$

where c_1 *and* c_2 *are real constants.*

(c) Finally determine values of c_1 *and* c_2 *such that* $x_1(0) = 2$ *and* $x_2(0) = 2$*:*

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} -2c_1 + 2c_2 + 4 \\ c_1 + c_2 - 1 \end{bmatrix}$$

This system of equations has the solution $c_1 = 2$ *and* $c_2 = 1$*, so the desired solution to* $(**)$ *is*

$$\mathbf{x} = \begin{bmatrix} -4 + 2e^{4t} + 4e^{-2t} \\ 2 + e^{4t} - e^{-2t} \end{bmatrix}.$$

13. Solve the initial value problem

$$y^{(3)} + 4y' = 2 + e^x; \quad y(0) = 5,\ y'(0) = \frac{1}{5}, \ \text{ and } \ y''(0) = 1.$$

Solution *First solve the homogeneous equation*

$$y^{(3)} + 4y' = 0. \tag{$*$}$$

The characteristic polynomial

$$r^3 + 4r = r(r^2 + 4)$$

has roots 0, $2i$, *and* $-2i$, *so the general solution to* $(*)$ *is*

$$y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x,$$

where c_1, c_2, *and* c_3 *are real constants.*

Next find particular solutions to the nonhomogeneous equations

$$y^{(3)} + 4y' = 2 \ \text{ and } \tag{1}$$

$$y^{(3)} + 4y' = e^x \tag{2}$$

Constants are solutions to $(*)$, *so the right hand side in* (1) *has duplication with the complementary solution* y_c. *A trial solution for* (1) *is therefore* $y_{p_1} = Ax$. *Now* (1) *yields*

$$4A = 2 \quad \text{so} \quad A = \tfrac{1}{2}.$$

Since e^x *is not a solution to* $(*)$, *a trial solution for* (2) *is* $y_{p_2} = Be^x$. *Now* (2) *yields*

$$Be^x + 4Be^x = e^x \quad \text{so} \quad B = \tfrac{1}{5}.$$

Thus, the general solution to $(**)$ *is*

$$y = y_c + y_{p_1} + y_{p_2} = c_1 + c_2 \cos 2x + c_3 \sin 2x + \tfrac{1}{2}x + \tfrac{1}{5}e^x,$$

where c_1, c_2, *and* c_3 *are real constants.*

The first and second derivatives are

$$\begin{aligned} y' &= -2c_2 \sin 2x + 2c_3 \cos 2x + \tfrac{1}{2} + \tfrac{1}{5}e^x \\ y'' &= -4c_2 \cos 2x - 4c_3 \sin 2x + \tfrac{1}{5}e^x \end{aligned}$$

so

$$\begin{aligned} 5 &= y(0) = c_1 + c_2 + \tfrac{1}{5} \\ \tfrac{1}{5} &= y'(0) = 2c_3 + \tfrac{1}{2} + \tfrac{1}{5} \\ 1 &= y''(0) = -4c_2 + \tfrac{1}{5} \end{aligned}$$

The second equation yields $c_3 = -\frac{1}{4}$; *the last one gives* $c_2 = -\frac{1}{5}$, *and from the first it then follows that* $c_1 = 5$. *Thus the solution is*

$$y = 5 - \tfrac{1}{5}\cos 2x - \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + \tfrac{1}{5}e^x.$$

14. Find the inverse Laplace transform of $F(s) = \frac{7}{s^2(s+1)(s+2)}$.

15. Show that $\mathcal{L}\{t^n e^{3t}\} = \frac{n}{s-3}\mathcal{L}\{t^{n-1}e^{3t}\}$.

16. Use the Laplace transform to solve the initial value problem $x'' + 8x' + 15x = 0$, $x(0) = 2$, and $x'(0) = -3$.

17. Find the general solution to the system of differential equations $\frac{dx}{dt} = x - 4y$ and $\frac{dx}{dt} = y$

18. Compute the determinant of the matrix

$$\begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$