QUIZ 4–SOLUTIONS

Work the following proplems. Write your solutions clearly.

(1) Find the general solution of the differential equation

$$y^{(3)} + 4y = 3x^3.$$

Solution *We first look at the homogeneous equation* $y^{(3)} + 4y = 0$. *The characteristic polynomial is given by* $r^3 + 4 = 0$. *We need to solve* $r^3 = -4$ *and so we need to find all the three roots of* $r^3 = -1$. *If* $r = \cos(\alpha) + i\sin(\alpha)$ *is a complex number verifing* $r^3 = -1$, *we obtain that*

$$\cos(3\alpha) + i\sin(3\alpha) = \cos(\pi) + i\sin(\pi).$$

In particular $3\alpha = \pi + 2k\pi$ *for all integer values of* k. *For* $k = 0$ *we get* $\alpha = \frac{\pi}{3}$, *for* $\alpha = 1$ *we get* $\alpha = \pi$, *and for* $k = 2$ *we obtain* $\alpha = \frac{5}{3}\pi$. *These values of* α *correpond to the different solutions of the equation* $r^3 = -1$. *So the solutions of the equation* $r^3 = -4$ *are*

$$r_1 = 4^{1/3}(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})), r_2 = 4^{1/3}(\cos(\pi) + i\sin(\pi)), r_3 = 4^{1/3}(\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}))$$

which means

$$r_1 = 4^{1/3}(\frac{1}{3} + i\frac{\sqrt{2}}{3}), r_2 = -4^{1/3}, r_3 = 4^{1/3}(\frac{1}{3} - i\frac{\sqrt{2}}{3}).$$

The general solution is given by

$$y(x) = Ae^{-4^{1/3}x} + e^{\frac{1}{3}4^{1/3}x}(C\cos(\frac{\sqrt{2}}{3}x) + D\sin(\frac{\sqrt{2}}{3}x).$$

To get a particular solution of the differential equation $y^{(3)} + 4y = 3x^3$, *we consider the linear combination* $f(x) = a + bx + cx^2 + dx^3$, *which third derivative is* $f^{(3)}(x) = 6d$. *We obtain*

$$6d + 4(a + bx + cx^2 + dx^3) = 3x^3$$

which yield to the following system of equations

$$6d + 4a = 0, 4b = 0, 4c = 0, 4d = 3,$$

with solutions $d = 3/4, a = -9/8.b = 0, c = 0$. *The solution is*

$$\boxed{y(x) = Ae^{-4^{1/3}x} + e^{\frac{1}{3}4^{1/3}x}(C\cos(\frac{\sqrt{2}}{3}x) + D\sin(\frac{\sqrt{2}}{3}x) - 9/8 + 3/4x^3}$$

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(2) Find all solutions for the equation $x^6 + 1 = 0$.

Solution *We need to find all solution of* $x^6 = -1$. *If* x *is a complex number such that* $x^6 = -1$, *we can write* $x = \cos(\alpha) + i\sin(\alpha)$. *This implies that* $\cos(6\alpha) + i\sin(6\alpha) = \cos(\pi) + i\sin(\pi)$ *and* $\alpha = \frac{\pi}{6} + \frac{2}{6}k\pi$, *for every integer value of* k. *In particular,*

$$k = 0 \quad \alpha = \tfrac{\pi}{6}$$
$$k = 1 \quad \alpha = \tfrac{3\pi}{6}$$
$$k = 2 \quad \alpha = \tfrac{5\pi}{6}$$
$$k = 3 \quad \alpha = \tfrac{7\pi}{6}$$
$$k = 4 \quad \alpha = \tfrac{9\pi}{6}$$
$$k = 5 \quad \alpha = \tfrac{11\pi}{6}$$

The six solutions are

$$x_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$
$$x_2 \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}$$
$$x_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$x_4 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$$
$$x_5 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$$
$$x_6 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$$