

Instructions: Make sure your exam has all six pages. You must justify your answer to receive credit for a given solution.

1. (15 points) Determine whether the existence and uniqueness theorem (Theorem 1 in section 1.3) guarantees existence of a solution to the initial value problem

$$\frac{dy}{dx} = \sqrt[3]{y-2}; \quad y(0) = 2.$$

Find an explicit solution to this initial value problem.

Solution

The function $f(y) = \sqrt[3]{y-2}$ is continuous at the point $(0, 2)$ but the partial derivative in y ($D_y(f(x, y)) = \frac{1}{3}(y-2)^{-2/3}$) is not continuous in the point $(0, 2)$. So the theorem is inconclusive. On the other hand the constant function $y = 2$ is a solution of the partial differential equation.

2. (10 points) A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine with a concentration of 1 lb salt per gallon enters the tank at the rate of 5 gal/s, and brine flows out of the tank at the rate of 3 gal/s.

Set up the appropriate differential equation.

Do not solve the differential equation but describe how you would use the solution to determine how much salt the tank will contain when it is full of brine.

Solution

Let $x(t)$ be the amount of salt in the tank at the time t . The equation would be:

$$\frac{dx}{dt} = 5 - 3 \frac{x(t)}{100 + 2t}.$$

I would solve then the differential equation and use the initial condition (at time $t = 0$ there are 50 lb of salt) to find the constant of integration. After that, I would find at what time t the volume $V(t) = 100 + 2t$ is equal to 400-gal. Finally, I would plug that value of time in the solution of the differential equation.

3. (15 points) Suppose that a population of a certain species consists initially of 100 elements. Assume that the death rate $\delta = 0$, and the birth rate $\beta = 0.0005P(t)$. Has this species a problem of extinction? **Solution**

Denote the population number by $P(t)$, we have

$$\frac{dP(t)}{dt} = 0.0005P^2(t).$$

The solution of such equation is $P(t) = \frac{1}{10^{-2} - 5 \cdot 10^{-4}t}$. Since, $\lim_{t \rightarrow 20} P(t) = \infty$,
the species does not have any problem with extinction.

4. (15 points) Find the critical points of the differential equation

$$\frac{dx}{dt} = 3x - x^2.$$

Then analyze the sign of the right-hand side of the differential equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation.

Solution

The critical points are the solution of the equation $3x - x^2 = 0$, $x = 0, x = 3$. The function $3x - x^2$ is negative for $x < 0$ and $x > 3$ while it is positive for $0 \leq x \leq 3$. This implies that the solution $x = 3$ is a stable solution while $x = 0$ is an unstable solution.

5. (10 points) Find the solution of the initial value problem

$$x^3 y' + 2x^2 y = 1, \quad y(1) = 1$$

Solution

Dividing by x^3 we obtain the equation $y' + 2\frac{y}{x} = \frac{1}{x^3}$. The $\rho(x) = e^{\int 2\frac{1}{x} dx} = x^2$. Multiplying both side by x^2 , we obtain the equation:

$$x^2 y(x) = \int \frac{1}{x} dx = \ln x + C.$$

The initial conditions give $C = 1$. So that $y(x) = \frac{\ln x + 1}{x^2}$

6. (10 points) Find the general solution of $(x^3 + \frac{y}{x})\frac{dx}{dy} + (y^2 + \ln x) = 0$.

Solution

$D_y(x^3 + \frac{y}{x}) = \frac{1}{x} = D_x(y^2 + \ln x)$, shows that the the differential equation is exact. An antiderivative:

$$F(x, y) = \int x^3 + \frac{y}{x} dx = \frac{1}{4}x^3 + y \ln x + g(y).$$

By taking derivative in y , we have

$$D_y(F(x, y)) = D_y(\frac{1}{4}x^3 + y \ln x + g(y)) = \ln x + g'(y) = y^2 + \ln x.$$

Which implies $g'(y) = y^2$, implies $g(y) = \frac{1}{3}y^3$, and therefore a general solution is

$$\text{ss } \frac{1}{4}x^3 + y \ln x + \frac{1}{3}y^3 + C = 0$$