TAKE HOME TEST 3

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

As before, you will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make conjectures, and back up your assertions with a proof or a counterexample. In writing your answers, use complete English sentences and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of numerical examples and try to find a pattern.

On this exam there are no choices. Everyone must do all five problems.

(1) The **Fibonacci sequence** is a sequence of integers $a_0, a_1, a_2, \ldots$ which is defined as follows: $a_0 = 1$, $a_1 = 1$ and for all $n \geqslant 2$, $a_n = a_{n-1} + a_{n-2}$. Thus, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, etc. Prove that for all $n \geqslant 0$ we have $a_n \leqslant (\frac{7}{4})^n$. (Hint: It is ok to assume both $P(k-1)$ and $P(k)$ are true when proving $P(k+1)$ is true.)

(2) The "E-Zone" is the set of even integers. Those integers which can not be written as a product of two even integers were called "E-primes". Every integer in the E-Zone can be factored as product of E-primes (you don't need to prove this, just write down few examples to convince yourself).

(a) Make a conjecture of the form "There are two different ways to factor an even integer n as product of E-primes if __________________." Then prove your conjecture.

(b) Make a conjecture of the form "There will be precisely *three* different ways to factor the even integer n as a product of E-primes if and only if __________________." Then prove your conjecture.

(3) Let a and b be positive integers, and write $a = p_1^{a_1} \cdots p_k^{a_k}$, $b = p_1^{b_1} \cdots p_k^{b_k}$ where each p_i is prime, $a_i \geq 0$, and $b_i \geq 0$. (In other words, we have the prime factorizations of a and b, but we allow 0's for exponents so that we can use the same list of primes for a and b. For example, if $a = 35$ and $b = 20$, we would write $a = (2^0)(5^1)(7^1)$ and $b = (2^2)(5^1)(7^0)$.)

(a) Explain why the set of positive divisors of a is exactly

$$\{p_1^{r_1} \cdots p_k^{r_k} \mid 0 \leq r_i \leq a_i \text{ for } i = 1, \ldots, k\}$$

(b) Find a formula (in terms of $a_1, \ldots, a_k$) for the number of positive divisors of a. (Hint: Find a way to count the number of elements in the set above.)

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(c) Find a formula for $\gcd(a, b)$ in terms of the prime factorizations of a and b as given above. Be sure to explain why your formula works.

(4) Prime integers which can be written in the form $2^n - 1$ for some integer n are called **Mersenne** primes after Marin Mersenne, the 17th century French mathematician who studied them. The first few Mersenne primes are $3 = 2^2 - 1$, $7 = 2^3 - 1$, and $31 = 2^5 - 1$. The world record largest known primes have nearly always been Mersenne primes. The current record for the largest prime number is $2^{24,036,583} - 1$, which was proved to be prime (using a computer, of course) in June 2004.

(a) Prove that if $2^n - 1$ is a Mersenne prime then n must also be prime. (Hint: it might be helpful to first find a factor of $a^m - 1$ where m is a positive integer.)

(b) Show by example that $2^p - 1$ may not be prime even if p is.

(5) For positive integer n, we define $n!$ to be $(n)(n-1)(n-2)\cdots 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. By convention, we define $0!$ to be 1. Now let $n \geqslant k$ be two non-negative integers. The *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

(a) Find $\binom{8}{2}$, $\binom{10}{4}$, $\binom{7}{5}$ and $\binom{12}{7}$.

(b) Prove that for all integers n and k such that $n \geqslant k \geqslant 1$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

(c) Prove that $\binom{n}{k}$ is an integer for all positive integers n and all integers k such that $0 \leqslant k \leqslant n$.

Statement of Sources: : Give a list of all people with whom you discussed the problems on this test. Also, if you used any references besides the class notes, list them as well.