TAKE HOME TEST 2

Solo – No Collaboration Allowed

On this test, your work is to be your own with no consultation with any other person (in this class or not) except for the instructor. Feel free to ask me any questions. I won't give you any answers to the problems but will be happy to try to clarify any confusion you may have, probably by asking you more questions. You may feel free to use any written references or books or even the internet, just no consultation with any persons.

As usual, you will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make conjectures, and back up your assertions with a proof or a counterexample. In writing your answers, use complete English sentences and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of numerical examples and try to find a pattern.

On this test you again have a choice. Do any **five** of the following six problems. You must decide which ones you think you can do best. If you do all six, I will count only the first five—not the best five.

Special Requests: : Please use standard $8\frac{1}{2}$ by 11 paper (lined or unlined, but please no "fringe"). Please start each problem at the top of a fresh sheet of paper (not on the back of a page you've already written on) and write your name on every page. Finally, please staple your pages together.

(1) Let a, b, c be non-zero integers. We know from high school algebra that an equation of the form $ax + by = c$ is the equation of a straight line in the plane. A point in the plane with both coordinates *integers* is called a **lattice point**. Note: You might want to use graph paper on this problem.

- (a) Draw a careful graph of the line $4x + 8y = 6$ in the plane. Does the graph pass through any lattice points? If so, list at least 4 such points (show them on the graph, too!) and explain, with reasons, how many it actually passes through. If not, explain why not.
- (b) Draw a careful graph of the line $6x + 3y = 8$ in the plane. Does the graph pass through any lattice points? If so, list at least 4 such points (show them on the graph, too!) and explain, with reasons, how many it actually passes through. If not, explain why not.
- (c) Describe the set of all lattice points (if any exist) on the graph of $6603x + 5680y = -213$.

(2) Fun with GCD's. In each of the following, a and b are positive integers.

- (a) Suppose $a = b + p$ where p is a prime number. What are the possible values of $\gcd(a, b)$? Do a few examples and form a conjecture. Then try to prove your conjecture.

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(b) Suppose $\gcd(a, b) = 1$. What is the least common multiple of a and b? Again, form a conjecture and then try to prove it. (The **least common multiple** of a and b is defined to be the smallest positive integer which is divisible by both a and b.)

(3) Observe that for $n = 1, 2, 3, 4, 5$, the values of the expression $n^2 - n + 41$ are 41, 43, 47, 53, 61 — all primes. Is this always true? In other words, is $n^2 - n + 41$ a prime integer for every integer n?

Here's another one: for any integer $k \geqslant 1$ let let p_k be the kth prime; i.e., $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$, etc. Is $(p_1)(p_2)\cdots(p_k) + 1$ always a prime number for any $k \geqslant 1$? For instance, when $k = 1$ we get $p_1 + 1 = 3$, which is prime. For $k = 2$ we have $(p_1)(p_2) + 1 = (2)(3) + 1 = 7$, which is also prime. For $k = 3$ we have $(p_1)(p_2)(p_3) + 1 = (2)(3)(5) + 1 = 31$, which is again prime. Will this always give us a prime number?

(4) Factoring big numbers is a very hard problem. But big numbers which have a very special form can often be completely factored into primes by using some algebraic tricks before doing trial divisions. Using the well-known factorizations of $a^2 - 1$, $a^3 - 1$, and $a^3 + 1$ to get a good start (look them up if you don't remember them), find the complete prime factorization of $7^{24} - 1$. (Note that $7^{24} - 1 = 191,581,231,380,566,414,400$ is a twenty-one digit number which you could factor by brute force trial divisions if your calculator carries enough digits – most don't – but it would be a very tedious task using only the trial division method.) Hint: Go as far as you can without using this, but I will tell you that 193 is a factor.

(5) *Twin primes* are pairs of consecutive odd integers $(p, p+2)$ which are both prime. For example, $(3, 5)$, $(5, 7)$, $(11, 13)$ are all pairs of twin primes. Give three more examples of twin primes. The Twin Prime Conjecture (still unresolved) claims that there are infinitely many pairs of twin primes.

Building on this idea, we can define a set of "triplet primes" to be three consecutive odd integers $(p, p+2, p+4)$ which are all primes. A quick scan of our list of primes less than 100 shows that there is exactly one set of triplet primes in that range: $(3, 5, 7)$. Are there any other sets of triplet primes (ever)? If so, give an example. If not, prove that there are no others.

Statement of Sources: As this was a "Solo" exam, you shouldn't have talked with anyone besides the instructor about this exam. Please write the following statement on your exam, and sign your name:

I have neither given nor received any help on this exam.

Also, if you used any references besides the class notes, list them as well.