

TAKE HOME TEST 1

Due: Friday, September 13

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

You will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counterexamples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Also remember that we are learning how to write math and a good start is to read the notes we wrote. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

Please use standard $8\frac{1}{2}$ by 11 paper (lined or unlined, but please no “fringe”). Please start each problem at the top of a fresh sheet of paper (not on the back of a page you’ve already written on) and write your name on every page. Finally, please staple your pages together.

- (1) Find the greatest common divisor of $a = 3591874$ and $b = 4577419$. If the greatest common divisor is d , write $d = ax + by$, for some integers x and y .
- (2) *Fun with greatest common divisors*
 - (a) Prove or disprove: Let a , b and c be any three integers then $\gcd(a, b) = \gcd(a, b + ac)$.
 - (b) Prove: Let a , b and c be any three integers. If $\gcd(a, c) = d$ and $\gcd(b, c) = 1$ then $\gcd(ab, c) = d$.
 - (c) Suppose a and b are integers such that $ax + by = 1$ for some integers x and y . Must $\gcd(a, b) = 1$? Similarly, if $ax + by = 2$ must $\gcd(a, b) = 2$?
- (3) The set of all odd positive integers can be divided into three classes, depending upon their remainders when divided by 6:
 - All those which are of the form $6n + 1$, namely 1, 7, 13, 19, etc. Let’s call these the **six-one** integers.
 - All those which are of the form $6n + 3$, namely 3, 9, 15, 21, etc. Let’s call these the **six-three** integers.
 - All those which are of the form $6n + 5$, namely 5, 11, 17, 23, etc. Let’s call these the **six-five** integers.

In class we proved that the product of two odd numbers is odd, the product of two even numbers is even, and the product of an odd and an even is even. What kinds of analogous statements can one make about the product of two six-one numbers, two six-three numbers, two six-five numbers, a six-one number and a six-three number, a six-one number and a six-five number, and a six-three number and a six-five number? (We know all these products will be odd, but will they be six-one numbers, six-three numbers, or six-five numbers?) Don't forget proofs!

- (4) Observe that for the first few positive integers n , the number $n^2 - n$ is a multiple of 2. For example, $1^2 - 1 = 0$, $2^2 - 2 = 2$, $3^2 - 3 = 6$, $4^2 - 4 = 12$, etc. Is this always true? In other words, is $n^2 - n$ always a multiple of 2?

Similarly, the numbers $1^3 - 1 = 0$, $2^3 - 2 = 6$, $3^3 - 3 = 24$, $4^3 - 4 = 60$, etc., all seem to be multiples of 3. Is this always true? In other words, is $n^3 - n$ always a multiple of 3?

Does this hold more generally? In other words, if n and k are positive integers, will $n^k - n$ always be a multiple of k ?

- (5) List all the possible (positive) divisors of the integers from 1 to 30. For example, 12 has the six divisors 1, 2, 3, 4, 6, and 12. You will notice that "most" positive integers have an even number of divisors, but a few have an odd number of divisors. Classify those positive integers which have an even number of divisors and those which have an odd number of divisors. Try to explain why your classification works in general.