THE JOY OF NUMBERS

1. A first look at primes

(1.1) **Definition.** An integer p is prime if $p \neq \pm 1$, the only factors are ± 1 and $\pm p$.

We wrote the first 100 integers, and we checked all the prime numbers.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

What we notice, is that after we cancell from the table all the multiples of $2, 3, 5, 7$, the only integers we are left we prime numbers. Jay conjectured the following:

(1.2) **Conjecture.** Every non prime integer less or equal then 100, has a prime factor which is $2, 3, 5$ or 7.

This is a great conjectured, and in fact we noticed that the table is actually a proof of the conjecture. But we also notice that there is another statement in the conjecture: *every non prime integer different from* ± 1, *is a multiple of a prime.* We decided to prove this right away.

(1.3) **Theorem.** *Every non prime integer different from* ± 1, *is a multiple of a prime.*

Proof. (Class). Assume by contradiction that there is a positive integer which is not prime, which is not equal to ± 1 and it is not a multiple of a prime integer. Let S be the set of all these integers, it is bounded below by zero. We already used the *axiom* which states that every set of integers, which is bounded below, has a minimum element. Denote by s the minimal element of S. Since s is an element of S, s is not a prime. By definition of a prime integer, there must be a factor different ± 1 and different from $\pm p$. We can write $s = k_1 k_2$, with $1 < k_1 < p$ and $1 < k_2 < p$.
Now here is the contradiction,l because of the following two statements:

(1) Sara says that k_1 cannot be in the set S: this is true because $k_1 < s$ and s is the minimal element of the set S.

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(2) Eric and Chris say that k_1 has to be in the set S: they are right. k_1 is not prime because it is a factor of s, and every element of S has not prime factors. Also k_1 cannot have any prime factor because every prime factor of k_1 is a prime factor of s, and every element in S has not prime factors. This says that k_1 is an element in S.

□

We also decided that there is nothing special about 100:

(1.4) **Proposition.** *Given a non prime integer n, it has a prime factor which is less or equal then $\sqrt{n}$.*

Proof. (Class). Let n be a non prime integer, Then as we proved in the Theorem there must be a prime factor p_1: $n = p_1 m$. Now either m is prime, or it has a prime factor p_2. So we can write $n = p_1 p_2 l$. If by contradiction $p_1 > \sqrt{n}$ and $p_2 > \sqrt{n}$, then $n = p_1 p_2 l. > \sqrt{n}\sqrt{n} l > nl > n$, which is obviously a contradiction. (How can a number be bigger then itsel?) □

We still need to give an answer to the exercise of last time.