THE JOY OF NUMBERS

1. The Money Problem II

Before going back to the king of Nobeh, we looked at two homework problems. We prove the following

(1.1) **Proposition.** *Let a and b be two integers. Then the following hold:*

(1) *Set $d = gcd(a, b)$ and let h and k be two integers such that $a = hd$ and $b = kd$. Then $gcd(h, k) = 1$.*
(2) *Assume that $0 < b \leq a$, and write $a = bq + r$ as in the Division Theorem. Then, $gcd(a, b) = gcd(a, r)$.*

Proof. For (1), assume by way of contradiction that $gcd(h, k) = x > 1$. Then there are two integers m and n such that $h = nx$ and $k = mx$. In particular, we obtain that $a = dh = dxn$ and $b = dk = dxm$, which shows that dx is a divisor of a and b. On the other hand, since $x > 1$, we have that $dx > d$, which is contradiction since d is the biggest among all common divisors of a and b.

For (2), again we show more: we show that the set of all common divisors of a and b is equal to the set of all common divisors of b and r. Let x be an integer such that $x|a$ and $x|b$, then $a = xh$ and $b = xk$ for some integers h and k. This implies that $r = a - bq = xh - qxk = x(h - qk)$, which shows that x is also a divisor of r.On the other hand from $a = bq + r$ we can see that every divisor of b and r is also a divisor of a. □

Another approach to the problem would be to try to prove that $gcd(ac, bc) = cgcd(a, b)$, for any given integers a, b and c.

We then whowed another computation of the *Back Substitution method*:

(1.2) (Emily) Find the $gcd(5336, 1541)$.

$$\begin{aligned} 5336 &= (3)(1541) + 713 \\ 1541 &= (2)(713) + 115 \\ 713 &= (6)115) + 23 \\ 115 &= (5)(23) + 0 \end{aligned}$$

So that $gcd(5336, 1541) = 23$. Now with *back substitution* we want to write 23 as *linear combination* of 5336 and 1541, i.e. we want to find integers x and y such that $23 = x5336 + y1541$.

$$\begin{aligned} 23 &= 713 - (6)(115) \\ &= 713 - 6(1541 - (2)(713)) \\ &= 13(713) - 6(1541) \\ &= 13(5336 - (3)(1541)) - 6(1541) \end{aligned}$$

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$$= 13(5336) - 45(1541)$$

so that $x = 13$ and $y = 45$.

We then talked about the Money problem a little bit and we decided that a mathematical formulation for the problem of the king is the following

(1.3) **Conjecture.** Let a, b and c be integers. Set $d = gcd(a, b)$. Then, c is a linear combination of a and b if and only if c is a multiple of d.

Or equivalently

Let a, b and c be integers. Set $d = gcd(a, b)$. Then, $c = xa + yb$ for some integers x and y if and only if $c = dh$ for some integer h.

We gave a proof of the *only if* statement:

*Proof. Only if statement in (**??**)* By assumption $c = dh$ for some integer h. By the Back-substitution method, we can find integers x' and y' such that $d = x'a + y'b$. Then $c = hd = h(x'a + y'b) = hx'a + hy'b = xa + yb$, where $x = hx'$ and $y = hy'$. □

2. On the equation $ax + by = c$

Assume that in the equation $ax + by = c$, c is a multiple of $gcd(a, b)$. Then we know that there exist solution of such equation. How many solutions are there? we observed that there is more than one solution in general. How can we write them all? We run some tests:

(2.1) **Example.** (Jay and his table) The equation $3x + 5y = 22$ has at least the following solutions:

x	-1	-6	-11	-16
y	5	8	11	14.

We conjectured that (x, y) is a solution if and only if $(x, y) = (-1 - 5n, 5 + 3n)$.

(2.2) **Example.** (Eric and his table) The equation $6x + 8y = 20$ has at least the following solutions:

x	2	6	10	14
y	1	-2	-5	-8.

We conjectured that (x, y) is a solution if and only if $3x + 4y = 10$. This is true because $3x + 4y = 10$ is obtained by $6x + 8y = 20$ dividing by 2 both side of the equality. Can we give an explicit formula for the integers x and y that are solutions?

(2.3) **Example.** (Kelsey and her table) The equation $3x + 2y = 10$ has at least the following solutions:

x	1	3	5	7
y	7	4	1	-2

Again, can we give an explicit formula for the integers x and y that are solutions?

(2.4) **Example.** (Alex and his table) The equation $12x + 15y = 10$ has at least the following solutions:

x	2	7	12	17
y	1	-3	-7	-11

We conjectured that (x, y) solution if and only if $(x, y) = (2 + 5n, 1 - 4n)$ for some $n \in \mathbb{Z}$.

(2.5) **Homework.** (1) Prove the *if* direction of Conjecture (**??**).
(2) Prove the conjecture of your table.
(3) For Kevin, prepare to present the proof of the Division Theorem.