THE JOY OF NUMBERS

1. Finishing the Fundamental Theorem of Arithmetic

We need to finish the proof of the Fundamental Theorem of Arithmetic, we need to prove the following

(1.1) **Proposition.** *Let $p, q_1, \ldots, q_n$ be prime numbers. Assume that $p \mid q_1 \cdots q_n$ then $p = q_i$, for some i.*

We will use what we proved long time ago:
Claim A: Let a, b, c be integers. If $gcd(a, b) = 1$ and $a \mid bc$ for some c, then $a \mid c$.

(1.2) **Remark.** Let p, q be two primes either $gcd(p, q) = 1$ or $p = q$.

We are now ready to prove the Proposition 1.1

Proof. We decided to prove the statement by induction on n:

$$P(n) : \text{if } p \mid q_1 \cdots q_n \text{ then } p = q_i \text{ for some integers } 1 \leq i \leq n.$$

$P(1)$ is true since if $q \mid p$, then $p = q$ since p and q are primes. Let assume $P(n)$ is true, we need to prove that the statement $P(n+1)$ holds. For this, assume that $p \mid q_1 \cdots q_n q_{n+1}$. Call $a = q_1 \cdots q_n$, this means that $p \mid a q_{n+1}$. By Remark 1.2, either $gcd(p, q_{n+1}) = 1$ or $p = q_{n+1}$. In the second case we are done, in the first case we can apply Claim A, to show that $p \mid a$. But $a \mid q_1 \cdots q_n$, then by the induction hyphothesis we know that $q = p_i$, for some $1 \leq i \leq n$. □

For next time, think about the following homework:

(1.3) **Exercise.**
1. Compute $911^{853}\%4$.
2. Check whether 892753 is divisible by 37.
3. Compute $3^{90}\%20$.
4. Let N be a number, write $N = d_1 \ldots d_n$, meaning that

$$N = d_1 + d_2 \cdot 10 + d_3 \cdot 10^2 + \cdots + d_n \cdot 10^{n-1}.$$

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