

# THE JOY OF NUMBERS

## 1. INDUCTION

We learned today one way to prove statements about the natural numbers. We denote by  $P(n)$  a statement about the natural number, the goal is to prove that the statement will hold true for all natural numbers bigger then a given number  $n_0$ . Examples of such statements are

- (1)  $n$  is an even number for all  $n \geq 3$ ;
- (2)  $n$  is an odd number for all  $n \geq 100$ ;
- (3)  $n$  is divisible by 2;
- (4)  $n$  is a square for  $n \geq 25$ ;
- (5)  $n^2 - n$  is divisible by 2.

One way to prove that  $P(n)$  is a true statement for every natural number  $n$ , we could just check that  $P(1)$  is true,  $P(2)$  is true,  $P(3)$  is true and on and on.... For example for (5), we could check that  $0^2 - 0$  is divisible by 2,  $1^2 - 1$  is divisible by 2,  $2^2 - 2$  is divisible by 2 and so on so forth.... The problem with this system is that the natural numbers are infinitely many and no matter how fast we are, it will take an infinite time.

The new strategy is the following: we check that  $P(n_0)$  holds true. Then we assume that  $P(n)$  holds true (we don't check if  $P(n)$  is true, we assume that it is true). If, by assuming  $P(n)$ , we can prove that also  $P(n + 1)$  holds true then we can conclude that  $P(n)$  is a true statement for all the natural numbers. This will take a finite time. The idea is that, if you know that once you take a step you can take another one, then after you take the first one you can virtually walk for ever.

Let's go over (3):  $n$  is divisible for all  $n \geq 0$ . The number 0 is divisible by any number, in particular is divisible by 2. Let's assume that  $P(n)$  is true, which means: let's assume that  $n$  is divisible by 2. Can we prove that  $n + 1$  is divisible by 2. By our assumption  $n$  is divisible by 2, therefore  $n = 2s$  for some natural number  $s$ . It follows that  $n + 1 = 2s + 1$ . If  $n + 1$  were divisible by 2, then there would exist an integer  $t$  such that  $n + 1 = 2t$ . This implies that  $2s + 1 = 2t$  and  $2(s - t) = 1$ , which is a contradiction. We cannot prove that  $P(n)$  implies  $P(n + 1)$  and in fact 1 is not divisible by 2. So statement (3) is not true for all the natural number.

Let's go over (5):  $n^2 - n$  is divisible for all  $n \geq 0$ .  $P(0)$  says that  $0^2 - 0$  is divisible by 2. Assume that  $P(n)$  is true. So we assume that  $n^2 - n$  is divisible by 2, which says that there is an integer  $t$  such that  $n^2 - n = 2t$ . With this only assumption, we want to prove that  $(n + 1)^2 - (n + 1)$  is divisible by 2.

$$(n + 1)^2 - (n + 1) = n^2 + 2n + 1 - n - 1 = n^2 - n + 2n = 2t + 2n = * 2(t + n)$$

which shows that  $(n + 1)^2 - (n + 1)$  is divisible by 2. We used the induction statement in \*.

Then we had fun trying to answer the following questions:

- (1) What is the sum of the first  $n$  numbers?

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- (2) What is the sum of the first  $n$  odd numbers?  
 (3) What is the sum of the square of the first  $n$  numbers?

We answered the first two questions in class and we left the third one as homework. For (1): We first look for a formula that gives  $1 + 2 + \dots + n$ . We run some experiments:

n 1+2+...+n  
 1 **1**  
 2 1+2=**3**  
 3 1+2+3=**6**  
 4 1+2+3+4=**10**  
 5 1+2+3+4+5=**15**  
 6 1+2+3+4+5+6=**21**

You guys are really fast and conjectured the following

(1.1) **Theorem.**  $1 + 2 + \dots + n = \frac{1}{2}n^2 + \frac{1}{2}n$ , for all natural number  $n \geq 1$ .

*Proof.* The statement  $P(n)$  is  $1 + 2 + \dots + n = \frac{1}{2}n^2 + \frac{1}{2}n$ .  $P(1)$  holds true since  $1 = \frac{1}{2}1^2 + \frac{1}{2}1$ . Assume that  $P(n)$  is true. So that  $1 + 2 + \dots + n = \frac{1}{2}n^2 + \frac{1}{2}n$ . We want to prove that  $(P(n+1))$  holds true:

$$1 + \dots + n + (n+1) =^* \frac{1}{2}n^2 + \frac{1}{2}n + n + 1 = \frac{1}{2}n^2 + n + \frac{1}{2} + \frac{1}{2}n + \frac{1}{2} = \frac{1}{2}(n+1)^2 + \frac{1}{2}(n+1).$$

where in \* we used  $P(n)$ . □

For (2) we also were trying to find a formula that gives the sum of  $1 + 3 + 5 + \dots + (2n - 1)$ . Again we run some experiments:

n 1+...+2n-1  
 1 **1**  
 2 1+3=**4**  
 3 1+3+5=**9**  
 4 1+3+5+7=**16**

this one was easy.

(1.2) **Theorem.**  $1 + 3 + \dots + (2n - 1) = n^2$

*Proof.*  $P(1)$  holds true because  $1 = 1^2$ . Assume  $P(n)$ , so we assume that  $1 + 3 + \dots + (2n - 1) = n^2$ . We want to prove  $P(n+1)$  holds true:

$$1 + 3 + \dots + (2n - 1) + (2n + 1) =^* n^2 + 2n + 1 = (n + 1)^2,$$

where we equality \* uses the induction hypothesis  $P(n)$ . □