EXERCISES

This is the last set of exercises. You can submit your solutions by hand by April 27th or electronically by May 2nd.

(1) (10 points) Let R a noetherian ring and $I \subset R$, $J \subset R$ be two ideals. Prove that $\overline{I} : J$ is integrally closed.

(2) (10 points) Let $(R, \mathbf{m}, \mathsf{k})$ be a local noetherian ring and let $I \subset R$ be an ideal. Prove that the following are equivalent:
   (a) I is nilpotent;
   (b) $\overline{I} = \overline{\mathbf{m}I}$;
   (c) $\overline{I} = \overline{(0)}$.

(3) (5 points) Let $(S, \mathbf{m}, \mathsf{k})$ be a local noetherian ring and $I \subset S$ be an integrally closed ideal. Let $J = (I :_S \mathbf{m})$. Let $R = S/I$ and assume that $\operatorname{depth}(R) = 0$. Show that for any R-module M, $\beta_i(M) \geq \beta_{i-1}(M)$, for all $i \geq 1$.

(4) (10 points)Given an ideal in a local ring $(S, \mathbf{m})$, define $o(I)$ to be the maximal integer n such that $I \subset \mathbf{m}^n$. Assume that S is regular and let $R = S/I$. Let $J = (I :_S \mathbf{m})$ and assume that $o(I) > o(J)$. Prove that for any R-module M, $\beta_i(M) \geq \beta_{i-1}(M)$, for all $i \geq 1$.

(5) (10 points) Let V be a valuation domain. Show that every finitely generated ideal is principal.

(6) (10 points) Let $I \subseteq R$ be an ideal such that $\cap I^n = 0$. Prove that if $gr_I(R)$ is a domain then R is a domain.

(7) (20 points) Let R be a local integral domain with field of fractions K and infinite residue field $\mathbf{k}$. Let $v_1, \ldots, v_n : K^* \to Z$ be discrete valuations of rank one such that $v_i(r) \geq 0$ for all $r \in R$ and $i = 1, \ldots, n$. Prove that there exists an $x \in I$ such that $v_i(x) = v_i(I)$, for all $i = 1, \ldots, n$.

(8) (10 points) Let R be an integral domain and v be a valuation on the field of fractions of R. Let Γ be the value group of v and $\gamma \in \Gamma$. Assume that for all $r \in R \setminus \{0\}$, $r \geq 0$. Prove that $I_\gamma = \{r \in R \mid v(r) \geq \gamma\} \cup \{0\}$ is an integrally closed ideal.

(9) (10 points) Assume R is a noetherian ring and $I \subset R$ is an ideal of R. Let $x \in R$. Show that $x \in \overline{I}$ if and only if there exists an integer n such that $r^m \in I^{m-n}$ for all $m > n$.

(10) (10 points) Let I be a monomial ideal in $k[x_1, \ldots, x_d]$. Prove that $m \in \overline{I}$ if and only if there exists an integer n such that r^n is the the product of n monomials in I.

(11) (10 points) Verify that the maps u_i in theorem 10.1.6 are valuations.

(12) (10 points) Compute the integral closure of the ideal $I = (x^3, x^2y, y^4, y^2z, z^3)$. Prove your result.

(13) (10 points) Let $I = (x^3, y^3, xy)$ be an ideal of $k[x, y]$. Compute all the Rees valuations of I. (Note we compute them in class you just need to go through the all process and check that you don't get anything new).