EXERCISES

This is the first homework assigment. You should turn in your solutions by the 28th of February. You need to turn in solutions of exercises for a total of 50 points. Have fun.

(1) (5 points) Let $\phi : R \to S$ be a faithfully flat extension of Noetherians rings. Prove that $\overline{IS} \cap R = \overline{I}$. (Assume that $\overline{I}$ is an ideal)
(2) (5 points) Let $R \subseteq S$ be a finite extension of Noetherian rings. Prove that $\overline{IS} \cap R = \overline{I}$.
(3) (2 points) Let R be a Noetherian ring and $I \subset R$ an ideal. Consider the set

$$I^c = \{r \in R \mid f(r) = 0, \text{ for some monic polynomial } f \text{ with coefficients in } I\}.$$

Prove that $I^c = \sqrt{I}$.
(4) (2 points) Let R be a UFD. Show that if I is a principal ideal then I is integrally closed.
(5) (5 points) Prove that the integral closure of $k[t^2, t^3]$ is $k[t]$.
(6) (5 points) Let $R \subseteq S$ be an integral extension. Let W be a multiplicatevely closed subset of R, $Q \in Spec(S)$ and $P = Q \cap R$. Show that S/Q is an integral extension of R/P and $W^{-1}R \subset W^{-1}S$ is an integral extension.
(7) (15 points) Let $R[t]$ be a polynomial ring in one variable over the ring R, let $J \subseteq R$ and $I \subset R[t]$ be ideals. Assume that I contains a monic polynomial and $I + JR[t] = R[t]$. Show that $J + I \cap R = R$.
(8) (10 points) Using Theorem 11, show that $\overline{\overline{I}} = \overline{I}$.
(9) (10 points) Let k be a field and $I \subset \mathsf{k}[x_1, \ldots, x_n]$ be a monomial ideal. Show that $\overline{I}$ is a monomial ideal.
(10) (10 points) Let R be a reduced noetherian ring with integral closure $\overline{R}$ inside the total ring of fractions.
  (a) Show that the conductor is in R.
  (b) Show that the conductor is the biggest common ideal of R and $\overline{R}$.
  (c) Show that the conductor contains a non-zero-divisor if and only if $\overline{R}$ is a finitely generated R-module over R.
(11) (10 points) Basic properties of reductions: Let R be a noetherian ring and $J_1 \subset I_1$ and $J_2 \subset I_2$ be reductions.
  (a) Prove that $J_1 + J_2 \subset I_1 + I_2$ is a reduction.
  (b) Prove that $J_1 J_2 \subset I_1 I_2$ is a reduction.
  (c) Prove that $J \subset I$ is a reduction if and only if $J(R/\mathbf{p}) \subset I(R/\mathbf{p})$ is a reduction for all $\mathbf{p} \in Min(R)$.
(12) (15 points) Let R be a one-dimensional ring whose completion is reduced. Show that R is Gorenstein if and only if $\mathbf{m}^{-1}$ is generated by two elements. (It is helpful to know that a ring is Gorenstein if and only if it is Cohen-Macaulay of type one.)
(13) (10 points) Let $(R, \mathbf{m})$ be a one dimensional local Noetherian ring with infinite residue field. Let $I \subset R$ be an $\mathbf{m}$-primary ideal and let $N \subset M$ be two finitely generated R-modules such that M/N has depth one. Let J be a minimal reduction of I such that $I^n = JI^{n-1}$ for all $n > h$. Prove that for all $n > h$ the following holds:

$$I^n M \cap N = I(I^{n-1} M \cap N).$$

(14) (10 points) Prove the statement in 13 without the assumption on the depth of M/N.

(15) (10 points) Let $0 \to K \to F \to M \to 0$ be an exact sequence, where F is a free R-module. Show that if $x_1, \ldots, x_d$ is a regular sequence on M and on R, then $(x_1, \ldots, x_d)^n F \cap K = (x_1, \ldots, x_d)^n K$ for all $n > 0$.

(16) (10 points) Let R be a Cohen-Macaulay ring. Let $I \subset R$ be a height d ideal an ideal generated by d elements. Prove that $x_1, \ldots, x_d$ is a regular sequence

(17) (10 points) Let $(R, \mathbf{m}, \mathsf{k})$ be a Cohen-Macaulay local Noetherian ring with infinite residue field. Let I be an $\mathbf{m}$-primary ideal and let J be a minimal reduction of I.
   (a) Prove that J is generated by a regular sequence on R.
   (b) Prove that any regular sequence on R is a regular sequence on a module which is a $d+1$-th syzygy.
   (c) Let $\mathbf{F} = \{F_i\}$ be a free resolution of a finitely generated R-module M, and let $M_i \subset F_{i-1}$ be the i-th syzygy. Prove that there exists an integer h such that $I^n F_{i-1} \cap M_i \subset I^{n-h} M_i$ for all $i > 0$ and for all $n > h$.

(18) (5 points) Let R be a noetherian ring and let $\mathbf{p}$ be a prime ideal of R. Prove that $\mathbf{p}$ is a minimal prime if and only if there exists a non-nilpotent element $c \in R$ such that $c\mathbf{p}^n = 0$, for some positive integer n.

(19) (20 points) Let $(R, \mathbf{m})$ be a one dimensional local ring. Prove that there exists an integer h such that for every $\mathbf{m}$-primary ideal I and for every reduction $J \subset I$ one has $I^{n+1} = JI^n$ for all $n \geq h$.

(20) (15 points) Give an example of a families of ideals $\{J_n \subset I_n\}_{n \geq 0}$, where $J_n \subset I_n$ is a reduction, such that the reduction numbers $r_{J_n}(I_n)$ are not bounded.

(21) (10 points) Let $(R, \mathbf{m})$ be an analytically unramified local ring. Prove that there exists an integer k such that $\overline{I^{n+k}} = I^{n-k}\overline{I^k}$ and $\overline{I^k}^n = \overline{I^{kn}}$.

(22) (10 points) Let $(R, \mathbf{m}, \mathsf{k})$ be a local noetherian ring with infinite residue field of dimension bigger or equal then two. Let M be a finitely generated R-module. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = \lambda(\hom_R(M, R/\mathbf{m}^n)$. Prove that there exists an integer h such that $f(n) < f(n+1)$ for $n > h$.

From the book:

(1) (10 points) page 89 exercise 4.11