On classical integral closure and integral closure relative to an Artinian module

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   - Rees’ Question
   - Multiplicities

2 The concept of Attached Primes
   - Secondary Representation
   - Attached Primes

3 Integral Closures
   - The Traditional One
   - Integral Closure Relative to an Artinian Module

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Let \((R, \mathfrak{m})\) be a commutative Noetherian local ring. Let \(a\) and \(b\) be ideals of \(R\).

**Question (Rees, 1990)**

What is the relationship between \(\overline{b}\), the classical Northcott-Rees integral closure of \(b\), and \(b^*(H)\), the integral closure of \(b\) relative to the Artinian \(R\)-module \(H\)?

Under what conditions are they equal?
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**Question (Rees, 1990)**

What is the relationship between \(\overline{b}\), the classical Northcott-Rees integral closure of \(b\), and \(b^{\ast(H)}\), the integral closure of \(b\) relative to the Artinian \(R\)-module \(H\)? Under what conditions are they equal?
Definition

Let $\mathfrak{a}$ be an $m$-primary ideal, and let $M$ be a finitely generated $R$-module. The *multiplicity of $\mathfrak{a}$ on $M$* is defined by

$$e(\mathfrak{a}; M) := \lim_{n \to \infty} \frac{r!}{n^r} \lambda(M/\mathfrak{a}^n M),$$

where $r = \dim R$.
Rees’s Theorem

Let \((R, m)\) be a formally equidimensional Noetherian local ring, and \(b \subseteq a\) be \(m\)-primary ideals of \(R\). Let \(M\) be a finitely generated \(R\)-module. Then \(\overline{a} = \overline{b}\) if and only if \(e(a; M) = e(b; M)\).
Definition

The Noetherian dimension of an Artinian $R$-module $H$ is defined by

$$\text{Ndim}_R(H) = \inf\{k \mid \exists x_1, \ldots, x_k \in \mathfrak{m} \text{ such that } \lambda(0 :_{H} x_1, \ldots, x_k) < \infty\}.$$ 

Definition

Let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of $R$ and $H$ an Artinian $R$-module such that $\lambda(0 :_{H} \mathfrak{a}) < \infty$.

The multiplicity of $\mathfrak{a}$ relative to $H$ is defined by

$$e'(\mathfrak{a}; H) = \lim_{n \to \infty} \frac{d!}{n^d} \lambda(0 :_{H} \mathfrak{a}^n),$$

where $d = \text{Ndim}_R(H)$. 

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Definition

The *Noetherian dimension* of an Artinian $R$-module $H$ is defined by

$$\text{Ndim}_R(H) = \inf \{ k \mid \exists x_1, \ldots, x_k \in m \text{ such that } \lambda(0:H x_1, \ldots, x_k) < \infty \}.$$ 

Definition

Let $\mathfrak{a} \subset m$ be an ideal of $R$ and $H$ an Artinian $R$-module such that $\lambda(0:H \mathfrak{a}) < \infty$.

The *multiplicity of $\mathfrak{a}$ relative to $H$* is defined by

$$e'(\mathfrak{a}; H) = \lim_{n \to \infty} \frac{d!}{n^d} \lambda(0:H \mathfrak{a}^n),$$

where $d = \text{Ndim}_R(H)$. 
Proposition

Let $\alpha$ and $\beta$ be ideals of $R$ such that $\beta \subseteq \alpha$ and $\lambda(0 :_H \beta) < \infty$. If $\alpha^*(H) = \beta^*(H)$, then $e'(\alpha; H) = e'(\beta; H)$. 
Question
Is there a theorem analogous to Rees’s Theorem, for Artinian modules?

Question
Is there a relationship between $e(a; M)$, the multiplicity of $a$ on $M$, and $e'(a; H)$, multiplicity of $a$ relative to $H$?
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A non-zero $R$-module $L$ is called **secondary** if its multiplication map, $\varphi_x : L \to L$, by any element $x \in R$, is either surjective or nilpotent.

A **secondary representation** for an $R$-module $L$ is a finite expression $L = L_1 + L_2 + \ldots + L_s$ where $L_i$ is secondary for $1 \leq i \leq s$.

We will say that $L$ is **representable** if such an expression exists.
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Attached Primes

Definition
Let $p$ be a prime ideal of $R$. We say that $p$ is an **attached prime** of $L$, if $p = (K :_R L) = \text{ann}_R(L/K)$ for some submodule $K$ of $L$.
The set of attached prime ideals of the $R$-module $L$ is denoted by $\text{Att}_R(L)$.

Proposition
If $L$ admits an reduced secondary representation $L = L_1 + L_2 + \ldots + L_s$, then $\text{Att}_R(L) = \{\sqrt{(0 :_R L_i)} : 1 \leq i \leq s\}$. 
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Attached Primes

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If $L$ admits an reduced secondary representation $L = L_1 + L_2 + \ldots + L_s$, then $\text{Att}_R(L) = \{ \sqrt{(0 :_R L_i)} : 1 \leq i \leq s \}$. 
Attached Primes

**Theorem**

If $L$ is an Artinian $R$-module, then $L$ admits a reduced secondary representation and so $\text{Att}_R(L)$ is a finite set.

**Theorem (Ooish, 1976)**

Let $(R, m)$ be a Noetherian local ring. If $M$ is a finitely generated $R$-module, then

$$\text{Ass}_R(M) = \text{Att}_R(D(M)).$$
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Theorem (Ooish, 1976)
Let $(R, m)$ be a Noetherian local ring. If $M$ is a finitely generated $R$-module, then

$$\text{Ass}_R(M) = \text{Att}_R(D(M)).$$
Let $M$ and $N$ be two $R$-modules, $\alpha \subseteq R$ an ideal and $i \in \mathbb{Z}$.

**Definition (Grothendieck, 1966)**

$$H^i_\alpha(N) := \lim_{\longrightarrow n} \Ext^i_R(R/\alpha^n, N),$$

is the $i$th **local cohomology module** with respect to $\alpha$.

**Definition (Herzog, 1970)**

$$H^i_\alpha(M, N) := \lim_{\longrightarrow n} \Ext^i_R(M/\alpha^nM, N),$$

is the $i$th **generalized local cohomology module** with respect to $\alpha$. 
Let $M$ and $N$ be two $R$-modules, $a \subseteq R$ an ideal and $i \in \mathbb{Z}$.

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Question (MacDonald and Sharp, 1971)

How to determine the set of attached primes of the (generalized) local cohomology modules?
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Definition (Northcott and Rees, 1954)

An element $x$ of $R$ is said to be **integrally dependent on** $\alpha$ if there exist elements $c_1, \ldots, c_n \in R$, with $c_i \in \alpha^i$ for $i = 1, \ldots, n$ such that

$$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n = 0.$$  

Moreover,

$$\bar{\alpha} := \{y \in R : y \text{ is integrally dependent on } \alpha\}$$

is an ideal of $R$, called the **integral closure of** $\alpha$. 

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Integral Closure

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is an ideal of $R$, called the **integral closure of** $\alpha$. 
Let $H$ an Artinian $R$-module.

**Definition (Sharp and Taherizadeh, 1988)**

An element $x$ of $R$ is said to be **integrally dependent on** $a$ relative to $H$ if there exists $n \in \mathbb{N}^*$ such that

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\left( 0 :_H \sum_{i=1}^{n} x^{n-i} a^i \right) \subseteq (0 :_H x^n).
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Moreover,

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a^*_H := \{ y \in R : y \text{ is integrally dependent on } a \text{ relative to } H \}
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is an ideal of $R$, called the **integral closure of** $a$ relative to $H$. 

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Integral Closure Relative to an Artinian Module

Let $H$ an Artinian $R$-module.

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An element $x$ of $R$ is said to be **integrally dependent on $\alpha$ relative to $H$** if there exists $n \in \mathbb{N}^*$ such that

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Integral Closure Relative to an Artinian Module

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What is the relationship between $\overline{b}$, the classical Northcott-Rees integral closure of $b$, and $b^*(H)$, the integral closure of $b$ relative to the Artinian $R$-module $H$?
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Answer to Rees’s Question

In 1990, Sharp, Tiras and Yassi responded to that question in a particular case: when \((R, m)\) is a quasi-unmixed local ring of dimension \(r\) and \(H = H^r_m(R)\) is the \(r\)th local cohomology module of \(R\) with respect to \(m\).

Our main result generalizes their result:

**Theorem**

Assume \((R, m)\) is a commutative Noetherian complete local ring. Let \(H\) an Artinian \(R\)-module. The following conditions are equivalent:

(i) \(\bar{a} = a^*(H)\) for every ideal \(a\) of \(R\);

(ii) \(\bar{0} = 0^*(H)\);

(iii) every minimal prime ideal of \(R\) belongs to \(\text{Att}_R(H)\).
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**Rees's Theorem**

Let \((R, m)\) be a formally equidimensional Noetherian local ring, and \(b \subseteq a\) be \(m\)-primary ideals of \(R\). Let \(M\) be a finitely generated \(R\)-module. Then \(\overline{a} = \overline{b}\) if and only if \(e(a; M) = e(b; M)\).

**Proposition**

Let \(a\) and \(b\) be ideals of \(R\) such that \(b \subseteq a\) and \(\lambda(0_H b) < \infty\). If \(a^*(H) = b^*(H)\), then \(e'(a; H) = e'(b; H)\).
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Example 1

Let \((R, \mathfrak{m})\) a Noetherian complete local ring.

Let \(M\) a finitely generated \(R\)-module such that \(\text{Supp}_R(M) = \text{Supp}_R(R)\).

Take \(H = D(M)\). Then \(\text{Att}_R(H) = \text{Ass}_R(M)\).

Also, every minimal prime ideal of \(R\) is in \(\text{Ass}_R(M)\).

Then every minimal prime ideal of \(R\) belongs to \(\text{Att}_R(H)\).

Therefore, for all ideal \(\mathfrak{b}\) of \(R\),

\[ \mathfrak{b} = \mathfrak{b}^\ast(H) \, . \]
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\overline{\mathfrak{b}} = \mathfrak{b}^*(H).
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Let \((R, \mathfrak{m})\) a Noetherian local ring.

Gu and Chu showed: if \(M\) and \(N\) are finitely generated \(R\)-modules such that \(pd(M) = d < \infty\) and \(\text{dim } N = n < \infty\), then

\[
\text{Att}_R(H^{d+n}_a(M, N)) = \{p \in \text{Ass}_R N : \text{cd}(a, M, R/p) = d + n\}.
\]

Let \(H := H^{n+d}_a(M, N)\) and suppose that \(\text{cd}(a, M, R/p) = d + n\) for all minimal prime ideals \(p\) of \(R\).

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\]
Example 2

Let \((R, \mathfrak{m})\) a Noetherian local ring. Gu and Chu showed: if \(M\) and \(N\) are finitely generated \(R\)-modules such that \(pd(M) = d < \infty\) and \(\dim N = n < \infty\), then

\[
\text{Att}_R(H^{d+n}_a(M, N)) = \{p \in \text{Ass}_RN : \text{cd}(a, M, R/p) = d + n\}.
\]

Let \(H := H^{n+d}_a(M, N)\) and suppose that \(\text{cd}(a, M, R/p) = d + n\) for all minimal prime ideals \(p\) of \(R\).

Then for all ideal \(b\) of \(R\)

\[
\overline{b} = b^{*}(H).
\]
Summary

1 Motivation
   - Rees’ Question
   - Multiplicities

2 The concept of Attached Primes
   - Secondary Representation
   - Attached Primes

3 Integral Closures
   - The Traditional One
   - Integral Closure Relative to an Artinian Module

4 Results
   - The Main Result
   - Conclusion
   - Examples

5 References
V. H. Jorge Perez and L. C. Merighe, *About a question of D. Rees on classical integral closure and integral closure relative to an Artinian module*.


References


Thank You