CHAPTER 6

Differentiation and Integration

6.7. Riemann-Stieltjes Integration

Riemann-Stieltjes integration is a generalization of Riemann integration. The essential change is to weight the intervals according to a new function, called the integrator. This weighting gives integrals that combine aspects of both discrete summations and the usual Riemann integral. The price to be paid for this increased generality is some careful work with partitions.

First, we introduce a notational convenience and then the crucial definition.

6.7.1. Definition. If P is a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, then we call an n-tuple of real numbers $Z = (z_1, z_2, \dots, z_n)$ an **evaluation sequence** for P if $z_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. Notice that z_i can equal z_{i+1}.

6.7.2. Definition. Consider bounded functions $f, g : [a, b] \to \mathbb{R}$. Given a partition of $[a, b]$, $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and an evaluation sequence, $Z = (z_1, z_2, \dots, z_n)$, for P, we define the **Riemann-Stieltjes sum**, or briefly, the **R-S sum**, for f with respect to g using P and Z, as

$$I_g(f, P, Z) = \sum_{j=1}^{n} f(z_j)[g(x_j) - g(x_{j-1})].$$

We say f is **Riemann-Stieltjes integrable** with respect to g, denoted $f \in \mathcal{R}(g)$, if, there is a number L so that for all $\varepsilon > 0$, there is a partition P_ε so that, for all partitions P with $P \supseteq P_\varepsilon$ and all evaluation sequences Z for P, we have

$$|I_g(f, P, Z) - L| < \varepsilon.$$

In this case, we say L is the **Riemann-Stieltjes integral** of f with respect to g, and write $\int_a^b f\,dg$ for L.

First off, notice that for $g(x) = x$, the Riemann-Stieltjes sum is the usual Riemann sum $I(f, P, Z)$ and the Riemann-Stieltjes integral is the usual Riemann integral. Second, if $g(x)$ is the Heaviside step function and f is continuous at 0, then some work will show that $\int_{-1}^{1} f\,dg = f(0)$.

Before developing the general properties of this integral, we compare this definition to the corresponding condition for the Riemann integral (Condition (4) in Theorem 6.3.8). The partition P_ε in the definition differs from the Riemann integral condition: we are *not* asking for a number δ so that for all partitions P with $\mathrm{mesh}(P) < \delta$, we have $|I_g(f, P, Z) - L| < \varepsilon$. Indeed, for a Riemann-Stieltjes integrable function, there may not be such a δ, as the next example shows.

Some textbooks define Riemann-Stieltjes integrability using this more restrictive condition. We leave it as an exercise to show that this "δ-condition" implies R-S integrability.

6.7.3. Examples.

(1) Define f on $[0,1]$ by $f(x) = 0$ for $x \in [0,1/2)$ and $f(x) = 1$ for $x \in [1/2,1]$. Define g to be the same as f, save only that $g(1/2) = 0$.

For any partition P of $[0,1]$ containing $1/2$, and any evaluation sequence Z for P, a bit of work shows $I_g(f,P,Z) = 1$. Thus, $\int_0^1 f\,dg$ exists (and equals 1, although we don't need this).

However, for any $\delta > 0$, there is always a partition Q with $\mathsf{mesh}(Q) < \delta$ so that $1/2 \notin Q$. Since f and g are both constant on $[0,1/2)$ and $(1/2,1]$, $f(x_j')[g(x_j)-g(x_{j-1})] = 0$ for any j with $[x_{j-1},x_j]$ contained in either of these two intervals. Thus, for any choice of evaluation sequence $Z = (z_1,\dots,z_n)$, we have $I_g(f,Q,X) = f(z_j)$, where j satisfies $x_{j-1} < 1/2 < x_j$. However, we can always choose z_j to be either less than $1/2$ or at least $1/2$ and so $I_g(f,Q,X)$ could be either value. Thus, no matter how small the mesh size of Q, we cannot control the value of the Riemann-Stieltjes sum in terms of the mesh size.

(2) Change the above example by making f the same as g. Then f is not Riemann-Stieltjes integrable with respect to g. To see this, consider any partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ and evaluation sequence $Z = (z_1,\dots,z_n)$ for P. Clearly there is some $k \in \{1,\dots,n\}$ so that $x_{k-1} \le 1/2 < x_k$. Then there is only one nonzero term in $I_g(f,P,X)$, which is $f(z_k)$. So if $z_k \le 1/2$, then $I_g(f,P,Z) = 0$ and if $z_k > 1/2$, then $I_g(f,P,Z) = 1$.

The key point is that no matter how we choose P, both of these possibilities can occur. Thus, for any number L, if we choose $\varepsilon = 1/2$, then if we had a suitable partition P_ε, we would have

$$|1 - L| < 1/2 \quad \text{and} \quad |0 - L| < 1/2.$$

Since there is no L that satisfies both inequalities, f is not Riemann-Stieltjes integrable with respect to g.

We collect together some basic facts about Riemann-Stieltjes integrals.

6.7.4. Theorem. *If $f_1 \in \mathcal{R}(g)$ and $f_2 \in \mathcal{R}(g)$ on $[a,b]$, and $c_1, c_2 \in \mathbb{R}$, then $c_1f_1 + c_2f_2 \in \mathcal{R}(g)$ on $[a,b]$ and*

$$\int_a^b c_1f_1 + c_2f_2\,dg = c_1\int_a^b f_1\,dg + c_2\int_a^b f_2\,dg.$$

If $f \in \mathcal{R}(g_1)$ and $f \in \mathcal{R}(g_2)$ on $[a,b]$, and $d_1, d_2 \in \mathbb{R}$, then $f \in \mathcal{R}(d_1g_1 + d_2g_2)$ on $[a,b]$ and

$$\int_a^b f\,d(d_1g_1 + d_2g_2) = d_1\int_a^b f\,dg_1 + d_2\int_a^b f\,dg_2.$$

Finally, if $a < b < c$ and f is R-S integrable with respect to g on both $[a,b]$ and $[b,c]$ then it is R-S integrable with respect to g on $[a,c]$ and

$$\int_a^c f\,dg = \int_a^b f\,dg + \int_b^c f\,dg.$$

PROOF. We prove the first result and leave the second and third as exercises.

Consider any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and evaluation sequence $X = (x'_1, \ldots, x'_n)$ for P. We have

$$\begin{aligned}
I_g(c_1f_1 + c_2f_2, P, X) &= \sum_{j=1}^{n}(c_1f_1(x'_j) + c_2f_2(x'_j))(g(x_j) - g(x_{j-1})) \\
&= c_1\sum_{j=1}^{n} f_1(x'_j)(g(x_j) - g(x_{j-1})) \\
&\qquad + c_2\sum_{j=1}^{n} f_2(x'_j)(g(x_j) - g(x_{j-1})) \\
&= c_1I_g(f_1, P, X) + c_2I_g(f_2, P, X)
\end{aligned}$$

Let $\varepsilon > 0$. Pick $\eta > 0$ so that $|c_1|\eta + |c_2|\eta < \varepsilon$. Since $f_1 \in R(g)$, there is P_1 so that, for all partitions P with $P \supseteq P_1$ and all evaluation sequences X for P, we have

$$\left|I_g(f_1, P, X) - \int_a^b f_1\,dg\right| < \eta.$$

Similarly, there is P_2 so that, for all partitions P with $P \supseteq P_1$ and all evaluation sequences X for P, we have

$$\left|I_g(f_2, P, X) - \int_a^b f_2\,dg\right| < \eta.$$

Thus, for any partition P with $P \supseteq P_1 \cup P_2$, and for any evaluation sequence X for P, we have

$$\begin{aligned}
&\left|I_g(c_1f_1 + c_2f_2, P, X) - \left(c_1\int_a^b f_1\,dg + c_2\int_a^b f_2\,dg\right)\right| \\
&\le |c_1|\left|I_g(f_1, P, X) - \int_a^b f_1\,dg\right| + |c_2|\left|I_g(f_2, P, X) - \int_a^b f_2\,dg\right|.
\end{aligned}$$

Since $P \supseteq P_1$ and $P \supseteq P_2$, it follows from the choices of P_1, P_2, and η, that this last expression is less than ε. ■

The following two theorems are the analogues of the substitution formula (see Equation (6.4.6) on page 167) and the integration by parts formula (see the equation at the bottom of page 166).

6.7.5. Theorem. *Suppose* $f, g : [a, b] \to \mathbb{R}$ *are bounded and* $\varphi : [c, d] \to [a, b]$ *is a strictly increasing, continuous function onto* $[a, b]$. *Let* $F = f \circ \varphi$ *and* $G = g \circ \varphi$. *If* $f \in \mathcal{R}(g)$ *on* $[a, b]$, *then* $F \in \mathcal{R}(G)$ *on* $[c, d]$ *and*

$$\int_c^d F\,dG = \int_a^b f\,dg.$$

PROOF. Consider any partition $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ of $[c, d]$ and evaluation sequence $X = (x'_1, \dots, x'_n)$ for P. Since φ is strictly increasing and onto, it is one-to-one, $\varphi(c) = a$, and $\varphi(d) = b$. Thus, $\varphi(P)$ is a partition of $[a, b]$ and $\varphi(X) = (\varphi(x'_1), \dots, \varphi(x'_n))$ is an evaluation sequence for $\varphi(P)$. Moreover,

$$I_G(F, P, X) = \sum_{j=1}^{n} f(\varphi(x'_j))[g(\varphi(x_j)) - g(\varphi(x_{j-1}))] = I_g(f, \varphi(P), \varphi(X)).$$

Let $\epsilon > 0$. As $f \in \mathcal{R}(g)$, there is a partition Q_ϵ so that for all partitions Q of $[a, b]$ with $Q \supseteq Q_\epsilon$ and all evaluations sequences Y for Q, we have

$$\left| I_g(f, Q, Y) - \int_a^b f\,dg \right| < \varepsilon.$$

Let $P_\varepsilon = \{\varphi^{-1}(x) : x \in Q_\varepsilon\}$. If P is a partition of $[c, d]$ with $P \supseteq P_\varepsilon$, then $\varphi(P) \supseteq Q_\varepsilon$. Using the above two displayed equations, we have, for all partitions P with $P \supseteq P_\varepsilon$ and all evaluation sequences X for P,

$$\left| I_G(F, P, X) - \int_a^b f\,dg \right| < \varepsilon.$$

By the definition, $F \in \mathcal{R}(G)$ on $[c, d]$ and $\int_a^b F\,dG = \int_a^b f\,dg$. ■

6.7.6. Theorem. *Let* $f, g : [a, b] \to \mathbb{R}$ *be bounded functions. If* $f \in \mathcal{R}(g)$ *on* $[a, b]$, *then* $g \in \mathcal{R}(f)$ *on* $[a, b]$ *and*

$$\int_a^b f\,dg + \int_a^b g\,df = g(b)f(b) - g(a)f(a).$$

PROOF. First, we need to relate $I_f(g, P, X)$ and $I_g(f, Q, Y)$ for a suitable choice of partition Q and evaluation sequence Y.

Consider any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ and evaluation sequence $X = (x'_1, \ldots, x'_n)$ for P. Then

$$I_f(g, P, X) = \sum_{j=1}^{n} g(x'_j)(f(x_j) - f(x_{j-1})).$$

We also have the telescoping sum

$$f(b)g(b) - f(a)g(a) = \sum_{j=1}^{n} f(x_j)g(x_j) - \sum_{j=1}^{n} f(x_{j-1})g(x_{j-1}).$$

Subtracting the first equation from the second, we have

$$\begin{aligned} \text{(6.7.7)} \quad & -I_f(g, P, X) + f(b)g(b) - f(a)g(a) \\ &= \sum_{j=1}^{n} f(x_j)(g(x_j) - g(x'_j)) + \sum_{j=1}^{n} f(x_{j-1})(g(x'_j) - g(x_{j-1})). \end{aligned}$$

Deleting any zero terms arising where $x'_j = x_{j-1}$ or $x'_j = x_j$, this last pair of summations is $I_g(f, P \cup X, Q)$. Here, Q is the evaluation sequence $(x_0, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, x_n)$ if $x'_j \in (x_{j-1}, x_j)$ for all j and has either an x_{j-1} or an x_j omitted if either $x'_j = x_{j-1}$ or $x'_j = x_j$ for some j.

Let $\epsilon > 0$. As $f \in \mathcal{R}(g)$, there is a partition P_ϵ so that for all partitions P of $[a, b]$ with $P \supseteq P_\epsilon$ and all evaluation sequences X for P, we have

$$\left| I_g(f, P, X) - \int_a^b f\, dg \right| < \varepsilon.$$

The crucial point is that if $P \supseteq P_\varepsilon$, then $P \cup X \supseteq P_\varepsilon$ and so we can conclude that

$$\left| I_g(f, P \cup X, Q) - \int_a^b f\, dg \right| < \varepsilon$$

where Q is constructed as in the last paragraph. Applying (6.7.7), we can conclude that

$$\left| I_f(g, P, X) - \left(-\int_a^b f\, dg + f(b)g(b) - f(a)g(a) \right) \right| < \varepsilon.$$

This shows that $g \in \mathcal{R}(f)$ on $[a, b]$ and that $\int_a^b g\, df = -\int_a^b f\, dg + f(b)g(b) - f(a)g(a)$. ■

Finally, we show that, for a smooth function g, the Riemann-Stieltjes integral with respect to g is really just a Riemann integral. In fact, the most interesting integrators will be the discontinuous ones.

6.7.8. Theorem. *If $f : [a,b] \to \mathbb{R}$ has $f \in \mathcal{R}(g)$, where $g : [a,b] \to \mathbb{R}$ is C^1, then fg' is Riemann integrable on $[a,b]$ and*

$$\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx.$$

PROOF. Let $\epsilon > 0$. By Exercise 6.7.I below, it suffices to find P_ϵ so that for all partitions P with $P \supseteq P_\epsilon$ and evaluation sequences X for P

$$\left| I(fg', P, X) - \int_a^b f\,dg \right| < \epsilon.$$

Consider any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[c,d]$ and evaluation sequence $X = (x'_1, \dots, x'_n)$ for P. Then

$$I(fg', P, X) = \sum_{j=1}^{n} f(x'_j)g'(x'_j)[x_j - x_{j-1}],$$

$$I_g(f, P, X) = \sum_{j=1}^{n} f(x'_j)[g(x_j) - g(x_{j-1})].$$

Comparing these two summations cries out for the application of the Mean Value Theorem to g on each interval $[x_{j-1}, x_j]$ to obtain t_j so that $g(x_j) - g(x_{j-1}) = g'(t_j)[x_j - x_{j-1}]$. Doing this, we have

$$I(fg', P, X) - I_g(f, P, X) = \sum_{j=1}^{n} f(x'_j)[g'(x'_j) - g(t_j)][x_j - x_{j-1}].$$

Let $\varepsilon > 0$. Since f is bounded, there M so that $|f(x)| \le M$ for all $x \in [a,b]$. As g' is uniformly continuous, there is $\delta_1 > 0$ so that, for all $x, y \in [a,b]$, if $|x - y| < \delta_1$, then

$$|g'(x) - g'(y)| \le \frac{\epsilon}{2M(b-a)}.$$

First, pick a partition P_1 with $\mathsf{mesh}(P_1) < \delta$. For any partition P with $P \supseteq P_1$ and any evaluation sequence X for P, we have

$$\begin{aligned} |I(fg', P, X) - I_g(f, P, X)| &\le \sum_{j=1}^{n} |f(x'_j)[g'(x'_j) - g'(t_j)][x_j - x_{j-1}]| \\ &\le \sum_{j=1}^{n} M \frac{\varepsilon}{2M(b-a)}[x_j - x_{j-1}] = \frac{\varepsilon}{2}. \end{aligned}$$

Since $f \in \mathcal{R}(g)$, there is P_2 so that, for any partition P with $P \supseteq P_2$ and any evaluation sequence X for P, we have

$$\left| I_g(f, P, X) - \int_a^b f\,dg \right| < \frac{\varepsilon}{2}.$$

Thus, for any partition P with $P \supseteq P_1 \cup P_2$ and for any evaluation sequence X, we have

$$\left| I(fg', P, X) - \int_a^b f\,dg \right| < \varepsilon.$$

By the exercise mentioned at the start of the proof, we're done. ■

We've avoided upper sums and lower sums so far, as they do not behave well for general integrators. Only if $g(x_j) - g(x_{j-1}) \geq 0$ are maximizing $f(x)$ and maximizing $f(x)[g(x_j) - g(x_{j-1})]$ equivalent problems. If the integrator is increasing, then upper and lower sums do have properties similar to upper and lower sums for Riemann integrals.

6.7.9. Definition. Let f and g be bounded functions on $[a, b]$. As usual, let $P = \{x_0 < x_1 < \cdots < x_n\}$ be a partition of $[a, b]$ and recall that

$$M_j(f, P) = \sup_{x_{j-1} \leq x \leq x_j} f(x) \qquad \text{and} \qquad m_j(f, P) = \inf_{x_{j-1} \leq x \leq x_j} f(x).$$

Define the **upper sum with respect to** g and the **lower sum with respect to** g to be

$$U_g(f, P) = \sum_{i=1}^{n} M_i(f, P)[g(x_i) - g(x_{i-1})]$$
$$L_g(f, P) = \sum_{i=1}^{n} m_i(f, P)[g(x_i) - g(x_{i-1})]$$

If we assume that g is increasing, then it is straightforward to verify that for any evaluation sequence X,

$$L_g(f, P) \leq I_g(f, P, X) \leq U_g(f, P). \tag{6.7.10}$$

This is not true if g is not increasing. It is possible for the above inequalities to be strict. For example, if $f(x) = x$ on $[0, 1)$ and $f(1) = 0$ and if $g(x) = x$, then $U_g(f, P) > I_g(f, P, X)$ for all choices of P and X.

As with the Riemann integral, we have

6.7.11. Refinement Lemma.

Let f, g be bounded functions on $[a, b]$ and P, R partitions of $[a, b]$. Assume that g is increasing. If R is a refinement of P, then

$$L_g(f, P) \leq L_g(f, R) \leq U_g(f, R) \leq U_g(f, P).$$

The proof is exactly the same as the Refinement Lemma for the Riemann integral. The key points are that if $x_{j-1}, x_j \in P$ and $[x_{j-1}, x_j] \cap R =$

$[t_k, \dots, t_l]$, then

$$g(x_j) - g(x_{j-1}) = \sum_{i=k+1}^{l} g(t_i) - g(t_{i-1})$$

and, for any i between $k+1$ and l,

$$m_j(f,P) \le m_i(f,R), \qquad M_i(f,R) \le M_j(f,P).$$

6.7.12. Corollary. *Let f, g be bounded functions on $[a,b]$ and assume that g is increasing. If P and Q are any two partitions of $[a,b]$,*

$$L_g(f,P) \le U_g(f,Q).$$

6.7.13. Definition. In keeping with the notation for Riemann integration, we define $L_g(f) = \sup_P L_g(f,P)$ and $U_g(f) = \inf_P U_g(f,P)$.

6.7.14. Riemann-Stieltjes Condition.
Let f, g be bounded functions on $[a,b]$ and assume that g is increasing on $[a,b]$. The following are equivalent:

(1) *f is Riemann-Stieltjes integrable with respect to g,*

(2) *$U_g(f) = L_g(f)$, and*

(3) *for each $\varepsilon > 0$, there is a partition P so that*

$$U_g(f,P) - L_g(f,P) < \varepsilon.$$

Moreover, if (2) holds, then common value equals $\int f\,dg$.

PROOF. The argument that (3) ⇒ (2) is immediate.

Suppose that (2) holds and let $L = L_g(f) = U_g(f)$. Let $\varepsilon > 0$. We can find two partitions P_1 and P_2 so that $U_g(f,P_1) < L+\varepsilon$ and $L_g(f,P_2) > L-\varepsilon$. Let $P_\epsilon = P_1 \cup P_2$ be their common refinement. By the Refinement Lemma,

$$L - \varepsilon < L_g(f,P_\epsilon) \le U_g(f,P_\epsilon) < L + \varepsilon.$$

Suppose that Q is any partition with $Q \supseteq P_\epsilon$ and X is any evaluation sequence for Q. Using (6.7.10) and the Refinement Lemma again,

$$L - \varepsilon < L_g(f,P_\epsilon) \le L_g(f,Q) \le I_g(f,Q,X) \le U_g(f,Q) \le U_g(f,P_\epsilon) < L + \varepsilon.$$

Thus, (1) holds and $L = \int f\,dg$

Next, suppose that (1) holds. Let $\varepsilon > 0$ and find a partition P_ε as in the definition of Riemann-Stieltjes integrability. We claim that $U_g(f,P_\varepsilon) - L_g(f,P_\varepsilon) < 2\varepsilon$, which suffices to prove (3). Assume $g(b) - g(a) > 0$, as otherwise $U_g(f,P_\varepsilon) = L_g(f,P_\varepsilon) = 0$.

Let $\eta > 0$. Suppose that $P_\varepsilon = \{x_0 < x_1 \dots < x_n\}$. For each j, choose points $s_j, t_j \in [x_{j-1}, x_j]$ so that $f(s_j) < m_j(f,P_\varepsilon) + \eta/[n(g(b) - g(a))]$ and

$f(t_j) > M_j(f, P_\varepsilon) - \eta/[n(g(b) - g(a))]$. Letting $S = (s_1, \dots, s_n)$ and $T = (t_0, \dots, t_n)$, we then have

$$\begin{aligned} I_g(f, P_\varepsilon, S) &= \sum_{j=1}^{n} f(s_j)[g(x_j) - g(x_{j-1})] \\ &< \sum_{j=1}^{n} \left(m_j(f, P_\varepsilon) + \frac{\eta}{n(g(b) - g(a))} \right) [g(x_j) - g(x_{j-1})] \end{aligned}$$

and using $g(x_j) - g(x_{j-1}) \le g(b) - g(a)$ (as g is increasing), we have

$$= L(f, P_\varepsilon) + \sum_{j=1}^{n} \frac{\eta}{n} = L(f, P_\varepsilon) + \eta.$$

Similarly, $I_g(f, P_\varepsilon, T) > U(f, P) - \eta$. Since $|I_g(f, P_\varepsilon, T) - I_g(f, P_\varepsilon, S)| < 2\varepsilon$, it follows that $U(f, P) - L(f, P) < 2\varepsilon + 2\eta$. Since η is arbitrary, this proves the claim. ■

An important change from the Riemann integral lies in bounds for integrals. A first guess might be that if f is bounded by M on an interval $[a, b]$, then $\int_a^b f\, dg$ would be bounded by $M(g(b) - g(a))$. While this is true if g is increasing on $[a, b]$, it is not true in general. For example, suppose g is identically zero on $[a, b]$ except for a subinterval in the interior, say $[c, d]$, where g is 1. If f is continuous at c and d, then it is easy to show that $\int_a^b f\, dg = f(c) - f(d)$. In particular, the integral may not be bounded above by $1(g(b) - g(a)) = 0$.

The problem is that g changes over the interval $[a, b]$ and we need to replace $g(b) - g(a)$ with the total change of g. The first step is to define this total change.

6.7.15. Definition. Given a function $f : [a, b] \to \mathbb{R}$ and a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < \dots < x_n = b\}$, then the **variation** of f over P is

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

6.7.16. Remarks. First, notice that if P is a partition of $[a, b]$ and Q is a partition of $[b, c]$, then, for a function $f : [a, c] \to \mathbb{R}$,

$$V(f, P \cup Q) = V(f, P) + V(f, Q).$$

Second, fix a partition P and a point $c \notin P$. Then, if $c \in [x_{j-1}, x_j]$, we have

$$|f(x_j) - f(x_{j-1})| \le |f(x_j) - f(c)| + |f(c) - f(x_{j-1})|.$$

By induction, it follows that if R is a refinement of P, then

$$V(f,P) \le V(f,R).$$

6.7.17. Definition. The **(total) variation** of f on $[a,b]$ is

$$V_a^b f = \sup\{V(f,P) : P \text{ a partition of } [a,b]\}.$$

We say f is of **bounded variation** on $[a,b]$ if $V_a^b f$ is finite.

6.7.18. Examples. If a function $f : [a,b] \to \mathbb{R}$ is increasing and $x \in [a,b]$, then a telescoping sum shows that $V_a^x f = f(x) - f(a)$. More generally, an easy estimate shows that if f is Lipschitz with constant C on $[a,b]$, then $V_a^b f \le C(b-a)$.

Define $g : [0,1/\pi] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that $V_0^{1/\pi} g = +\infty$. Fix $n > 1$ and let P_n be the partition

$$\{0, \frac{2}{\pi 2n}, \frac{2}{\pi(2n-1)}, \cdots, \frac{2}{3\pi}, \frac{2}{2\pi} = 1/\pi\}.$$

As g takes successive values of $0, -1, 0, 1, 0, \dots,$ for each interval $[x_{j-1}, x_j]$, $|g(x_j) - g(x_{j-1})| = 1$. Thus, $V(f,P_n) = 2n$ and $V_0^{1/\pi} f = +\infty$.

Define $h : [0,1/\pi] \to \mathbb{R}$ by

$$h(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Even this h is not of bounded variation. Using the partition P above, some work will show that $V(f,P)$ is a multiple of $\sum_{k=1}^n 1/k$, which diverges.

Finally, we define $k : [0,1/\pi] \to \mathbb{R}$ by

$$k(x) = \begin{cases} x^2\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Observe that k has a bounded derivative, $|k'(x)| = |2x\sin(1/x) - \cos(1/x)| \le 3$, and so k is Lipschitz.

6.7.19. Lemma. *If $a < b < c$ and $f : [a,c] \to \mathbb{R}$ is given, then*

$$V_a^c f = V_a^b f + V_b^c f.$$

PROOF. Let P be a partition of $[a,c]$. Let $P' = P \cup \{b\}$, $P_1 = P' \cap [a,b]$, and $P_2 = P' \cap [b,c]$. Then P_1 is a partition of $[a,b]$, P_2 is a partition of $[b,c]$, and $P' = P_1 \cup P_2$. Thus,

$$V(f,P) \le V(f,P') = V(f,P_1) + V(f,P_2) \le V_a^b f + V_b^c f.$$

Taking the supremum over all partitions P, we have $V_a^c f \le V_a^b f + V_b^c f$.

For the reverse inequality, let $\epsilon > 0$. There is a partition P_1 of $[a, b]$ so that $V(f, P_1) \ge V_a^b f - \epsilon/2$. and a partition P_2 of $[b, c]$ so that $V(f, P_2) \ge V_b^c f - \epsilon/2$. Thus, if $P = P_1 \cup P_2$, then the remark above shows that

$$V_a^c f \ge V(f, P) = V(f, P_1) + V(f, P_2) \ge V_a^b f + V_b^c f - \epsilon$$

As ϵ is arbitrary, this shows $V_a^c f \ge V_a^b f + V_b^c f$. ■

The key result about functions of bounded variation is the following characterization.

6.7.20. Theorem. *If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.*

Proof. Define

$$g(x) = \frac{1}{2}\left[f(a) + f(x) + V_a^x f\right]$$
$$h(x) = \frac{1}{2}\left[f(a) - f(x) + V_a^x f\right]$$

Clearly, $f(x) = g(x) - h(x)$. Notice that if $x < y$, then

$$g(y) - g(x) = \frac{1}{2}\left[f(y) - f(x) + V_x^y f\right]$$

If $P = \{x, y\}$, then $V_x^y f \ge V(f, P) = |f(y) - f(x)|$. In particular,

$$V_x^y f + f(y) - f(x) \ge 0$$

and so $g(y) - g(x) \ge 0$. Similarly, h is increasing. ■

With these results in hand, we can now bound Riemann-Stieltjes integrals.

6.7.21. Theorem. *If $f : [a, b] \to \mathbb{R}$ is bounded by M, $g : [a, b] \to \mathbb{R}$ is of bounded variation and $f \in \mathcal{R}(g)$ on $[a, b]$, then*

$$\left|\int_a^b f\,dg\right| \le M \cdot V_a^b g.$$

Proof. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and any evaluation sequence X for P, we have

$$|I_g(f, P, X)| \le \sum_{i=1}^{n} M|g(x_i) - g(x_{i-1})| \le M \cdot V_a^b g.$$

Choosing a sequence of partitions P_n and evaluation sequences X_n so that $I_g(f, P_n, X_n)$ converges to $\int_a^b f\,dg$, we have the result. ■

6.7.22. Theorem. *If $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is increasing, then $f \in \mathcal{R}(g)$ on $[a,b]$.*

PROOF. We assume $g(b) - g(a) > 0$, as otherwise g is constant and then every function is in $\mathcal{R}(g)$. Fix $\epsilon > 0$. As f is continuous and $[a,b]$ compact, f is uniformly continuous. Thus, there is $\delta > 0$ so that x, y in $[a,b]$ and $|x - y| < \delta$ implies

$$|f(x) - f(y)| \le \epsilon' = \frac{\epsilon}{g(b) - g(a)}.$$

Let P be a partition of $[a,b]$ with $\mathsf{mesh}(P) < \delta$. Then $M_j(f,P) - m_j(f,P) \le \epsilon'$. If $P = \{a = x_0 < x_1 < \dots < x_n = b\}$, then

$$U_g(f,P) - L_g(f,P) \le \sum_{j=1}^{n} \epsilon'[g(x_j) - g(x_{j-1})] = \frac{\epsilon(g(b) - g(a))}{g(b) - g(a)} = \epsilon,$$

where we have used the fact g is increasing to obtain a telescoping sum. By the R-S condition, $f \in \mathcal{R}(g)$. ■

6.7.23. Corollary. *If $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is of bounded variation, then $f \in \mathcal{R}(g)$ on $[a,b]$.*

PROOF. Using Theorem 6.7.20, $g = c - d$ where c, d are increasing functions on $[a,b]$. By the previous theorem, $f \in \mathcal{R}(c)$ and $f \in \mathcal{R}(d)$ so by Theorem 6.7.4, $f \in \mathcal{R}(c - g) = \mathcal{R}(g)$. ■

Exercises for Section 6.7

A. Define $h : [0, 1/\pi] \to \mathbb{R}$ by

$$h(x) = \begin{cases} x \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Write out carefully the argument above that h is not of bounded variation on $[0, 1/\pi]$. If $k(x) = xh(x)$, show that k is of bounded variation on $[0, 1/\pi]$.

B. Show that if $f : [a,b] \to \mathbb{R}$ is continuous and of bounded variation, then $f = g - h$ where g, h are continuous and increasing on $[a,b]$. HINT: Show that for f continuous, $k(x) = V_a^x f$ is continuous.

C. Show that if $g : [a,b] \to \mathbb{R}$ is C^1, then g is of bounded variation. HINT: Apply the Mean Value Theorem to $V(g,P)$ and observe that g' is bounded.

D. Suppose that $g : [a,b] \to \mathbb{R}$ is of bounded variation and that $\varphi : [c,d] \to [a,b]$ is increasing, continuous, and onto. Show that $G = g \circ \varphi$ is of bounded variation and find a formula relating $V_a^b g$ and $V_c^d G$.

E. Suppose that $f(x) = \mathsf{sign}(x)$. Show that if g is the Heaviside step function, then $\int_{-1}^{1} f \, dg$ does not exist.

F. Suppose that f, g are bounded on $[a, b]$ and g is increasing on $[a, b]$. For any partition P and any $\varepsilon > 0$, there are evaluation sequences X and Y, so that $I_g(f, P, X) > U_g(f, P) - \varepsilon$ and $I_g(f, P, Y) < L_g(f, P) + \varepsilon$.

G. Suppose $f : [a, b] \to \mathbb{R}$ is continuous except at finitely many points and g is continuous at those points and of bounded variation. Show that $f \in \mathcal{R}(g)$.

H. Suppose $\sum_{i \geq 1} r_i$ is a convergent series of positive numbers and x_i is a sequence of real numbers in $(a, b]$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If H is the Heaviside step function and $g(x) = \sum_{i=1}^{\infty} r_i H(x - x_i)$, show that g is well-defined and that

$$\int_a^b f \, dg = \sum_{i=1}^{\infty} r_i f(x_i).$$

I. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann-Stieltjes integrable with respect to $g(x) = x$. Prove directly from the definition of Riemann integrable, i.e. find a $\delta > 0$ as in the definition, that f is Riemann integrable.

J. Suppose that $g : [a, b] \to \mathbb{R}$ is of bounded variation and let $h(x) = V_a^x g$ for $x \in [a, b]$. If $f : [a, b] \to \mathbb{R}$ satisfies $f \in \mathcal{R}(g)$, then $f \in \mathcal{R}(h)$.

K. Given a function $g : [a, b] \to \mathbb{R}$, so that, for each continuous $f : [a, b] \to \mathbb{R}$, $\int f \, dg$ exists, then show that

$$\sup \left\{ \int f \, dg : f \in C([a, b]), \|f\|_\infty \leq 1 \right\} = V_a^b(g).$$

L. Define $g : [0, 1/\pi] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^{1/2} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Find a continuous function $f : [0, 1/\pi] \to \mathbb{R}$ so that $f \notin \mathcal{R}(g)$. Thus, in Corollary 6.7.23, we cannot change 'g of bounded variation' to 'g continuous'.

M. For h as in Examples 6.7.18, is it true that any continuous function $f : [0, 1/\pi] \to \mathbb{R}$, we have $f \in \mathcal{R}(h)$?