(1) Jan '06 # 2
  (a) Let Θ be a collection of pairwise disjoint open intervals of $\mathbb{R}$. Show that Θ is at most countable.
  (b) Show that the set of all increasing sequences of natural numbers (i.e. sequences $(n_1, n_2, ...)$ with $n_k \in \mathbb{N}$ and $n_{k+1} \geq n_k$ for all $k \in \mathbb{N}$) is uncountable.

(2) June '04 #6 Let $f : (0,1) \to \mathbb{R}$ be a function
  (a) Give the technical definition (the $\epsilon - \delta$ definition) of $\lim_{x \to a} f(x) = L$ where $a \in (0,1)$ and $L \in \mathbb{R}$.
  (b) Use the definition only to show that $\lim_{x \to a} \frac{x^2}{1-x} = \frac{a^2}{1-a}$ for every $a \in (0,1)$.

(3) June '02 #1
  (a) Let $\{a_k\}$ be a sequence of real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is convergent, and that the series $\sum_{k=1}^{\infty} a_k^2$ is divergent. Prove that the series $\sum_{k=1}^{\infty} a_k$ does not converge absolutely.
  (b) Consider the series $\sum_{k=1}^{\infty} \frac{1}{1+z^k}$, where $z \in \mathbb{C}$. Prove that the series diverges for all $|z| \leq 1$; and the series converges absolutely for all $|z| > 1$.

(4) Jan '03 #1 Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, differentiable on $(0,1)$, $f(0) = f(1) = 0$ and there is $x \in (0,1)$ with $f(x) = 1$. Prove that there is some $c \in (0,1)$ with $|f'(c)| > 2$.

(5) June '02 #3
  (a) Give a careful $\epsilon - \delta$ proof that $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$
  (b) Assume that f is differentiable at a. Evaluate

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a} \qquad (n \in \mathbb{N})$$