(1) Let $\epsilon > 0$ be given and find $\delta > 0$ s.t.

$$\rho(x, y) < \delta \ (x, y \in B) \Rightarrow \sigma(g(x), g(y)) < \epsilon$$

Since $f_n \to f$ unif on A, find N s.t.

$$\rho(f_n(x), f(x)) < \delta \quad \text{for all } n \geq N \quad \forall x \in A$$

Thus if $n \geq N$ then for any $x \in A$ $\sigma(g(f_n(x)), g(f(x))) < \epsilon$

ie// $g \circ f_n \to g \circ f$ uniformly

(2) 1) Let $a > 1$ be fixed.

Given $\epsilon > 0$, use the fact that $a^n \to \infty$ to find N s.t.

$$n \geq N \Rightarrow 1/a^n < \epsilon.$$

Then when $n \geq N$, $|1/x^n| \leq 1/a^n < \epsilon$ for all $x \in [a, +\infty)$

ie// $1/x^n \to 0$ uniformly on $[a, \infty)$

2) If $x \in (1, \infty)$ then $x > 1$ and so $1/x^n \to 0$

So if f_n were to converge uniformly to anything on $(1, \infty)$, its uniform limit would have to be zero.

Suppose $f_n \to 0$ uniformly on $(1, \infty)$.

Let $\epsilon = 1/2$ Then $\exists N$ s.t. $|f_n(x)| < 1/2$ for all $x > 1$ and all $n \geq N$

In particular $|1/x^N| < 1/2$ for all $x > 1$

But $\lim_{x \to 1^+} 1/x^N = 1$, contradicting this.

Thus f_n doesnt converge uniformly on $(1, \infty)$.

(3) Let $\epsilon > 0$ be given. WLOG suppose $0 < \epsilon < 1$

If $0 \le x < \epsilon$ then $|x(1-x)^n| < |x| < \epsilon$

Consider $\epsilon \le x \le 1$. Then $|x(1-x)^n| \le |1-x|^n \le (1-\epsilon)^n$

Since $0 < 1-\epsilon < 1$, $(1-\epsilon)^n \to 0$, so there is an N s.t. $n \ge N \Rightarrow (1-\epsilon)^n < \epsilon$.

Thus, if $n \ge N$, for each $x \in [0,1]$,

either $0 \le x < \epsilon$, in which case $|x(1-x)^n| < \epsilon$

or else $\epsilon \le x \le 1$, in which case $|x(1-x)^n| \le (1-\epsilon)^n < \epsilon$.

(can also use calculus to find the global max of $x(1-x)^n$ on $[0,1]$)

(4) No. Consider $f_n(x) = g_n(x) = x + 1/n$ on $\mathbb{R}$. Let $f(x) = g(x) = x$

Clearly $|f_n(x) - f(x)| = |g_n(x) - g(x)| = 1/n$

so $f_n \xrightarrow{\text{unif}} f$ and $g_n \xrightarrow{\text{unif.}} g$

However $f_n(x) g_n(x) = x^2 + \frac{2x}{n} + \frac{1}{n^2}$

Clearly $f_n(x) g_n(x) \to x^2$ pointwise so if $f_n g_n$ is to converge uniformly, this must be its limit.

Suppose $f_n(x) g_n(x) \longrightarrow x^2$ unif on $\mathbb{R}$.

Let $\epsilon = 1$. Find N s.t. $n \ge N \Rightarrow |f_n(x) g_n(x) - x^2| < 1$ for all $x \in \mathbb{R}$

But even for $n = N$

$$|f_N(x) g_N(x) - x^2| = \left| \frac{2x}{N} + \frac{1}{N^2} \right|$$

So take $x = N$ and

$$|f_N(x) g_N(x) - x^2| = 2 + 1/N^2 > 1.$$

Contrad.