# Math 826 Homework 4
## Assigned: 3/4/2009 (Ἐξελαύνω Day), Due: 3/16/2009

1. Let $(X, d)$ be a metric space. Prove the following:
   (a) $B_r(x)$ is open, for any fixed $x \in X$ and $r > 0$.
   (b) Any finite subset $F$ of $X$ is closed.
   (c) For fixed $x_0$ the function $f(x) := d(x, x_0)$ is continuous.
   (d) $\{x \in X \ : \ d(x, x_0) \leq r\}$ is closed. Is this set necessarily equal to $\overline{B_r(x_0)}$?

2. Prove Proposition 8.12: Say $x$ is a **closure point** if $S \subseteq X$ is there is a sequence $x_n \in S$ which converges to $x$. Prove that $\overline{S}$ concides with the set of all closure points of $S$.

3. Prove Proposition 8.13, that $S^\circ$ is the largest open set contained in $S$.

4. Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces and let $S \subseteq X$. Suppose $f : S \longrightarrow Y$ is *uniformly* continuous and $(Y, \sigma)$ is *complete*. Prove that $f$ has a unique extension to a uniformly continuous function $\overline{f} : \overline{S} \rightarrow Y$. (Hints: If $x \in \overline{S}$, find $x_n \longrightarrow x$ and define $\overline{f}(x) := \lim f(x_n)$. You need to prove that this is well-defined, extends $f$, and is continuous.)

5. Let $(X, d)$ be a metric space¿ For $x \in X$ and $S \subseteq X$, define

$$d(x, S) := \inf\{d(x, s) \ : \ s \in S\}$$

Show that $d(x, S) = 0$ iff $x \in \overline{S}$. Next, for $S, T \subseteq X$, define

$$d(S, T) := \sup\{d(s, T) \ : \ s \in S\}$$

and define

$$d_H S, T := \max\{d(S, T), d(T, S)\}$$

Prove that $d_H(S, T) = 0$ iff $S$ and $T$ have the same closures.