Math 826 Homework 3
Assigned: 2/11/2009, Due: 2/20/2009

1. Let f be a bounded function on $[a, b]$ and $g(x) := x$. Prove that f is Riemann integrable on $[a, b]$ if and only if it is Riemann-Stieltjes integrable with respect to g on $[a, b]$. Further prove that in this case

$$\int_a^b f(x)\,dx = \int_a^b f(x)\,dg(x)$$

2. Let $H(x)$ be the Heaviside function. Does $\int_{-1}^{1} H(x)\,dH(x)$ exist? Either evaluate it, or prove that it does not exist.

3. For functions f and g on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, prove that $V_a^b(\alpha f + \beta g) \leq |\alpha| V_a^b f + |\beta| V_a^b g$.

4. Read 6.7.14 (Riemann-Stieltjes Condition) and 6.7.22 (Continuous functions) from Davidson and Donsig's notes. (You don't need to turn anything in for this!)

5. Let $g(x)$ be defined by

$$g(x) := \begin{cases} x^{1/2}\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Construct a continuous function $f(x)$ on $[0, 1/\pi]$ so that f is **not** Riemann-Stieltjes integrable with respect to g. (This shows that in 6.7.22, "bounded variation" cannot be replaced with "continuous".)