1. Consider a function $f$ that is differentiable on the open interval $(A, B)$ and let $a < b$ lie in this interval.

(a) Suppose first that $f'(a) < 0 < f'(b)$, and show that there are $a < a' < b' < b$ such that $f(a') < f(a)$ and $f(b') < f(b)$. Hence, considering the minimum of $f$ on $[a, b]$, show that there is an $a < c < b$ with $f'(c) = 0$.

(b) Next, prove the general case: Suppose that $f'(a) < L < f'(b)$ and prove that there is a $c \in (a, b)$ such that $f'(c) = L$. This result is Darboux’s Theorem.

2. Let $f$ be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Suppose that $f'$ is strictly increasing and that $f(0) = 0$. Prove that $g(x) := f(x)/x$ is strictly increasing on $(0, \infty)$.

3. Suppose that $f'$ exists and is continuous on $[0, 1]$. Suppose further than the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ converges. Prove that $f(0) = f'(0) = 0$. 