Math 826 Homework 1
Assigned: 1/12/2009, Due: 1/19/2009

1. Consider a functions f that is differentiable on the open interval (A, B) and let $a < b$ lie in this interval.

   (a) Suppose first that $f'(a) < 0 < f'(b)$, and show that there are $a < a' < b' < b$ such that $f(a') < f(a)$ and $f(b') < f(b)$. Hence, considering the minimum of f on $[a, b]$, show that there is a $a < c < b$ with $f'(c) = 0$.

   (b) Next, prove the general case: Suppose that $f'(a) < L < f'(b)$ and prove that there is a $c \in (a, b)$ such that $f'(c) = L$. This result is **Darboux's Theorem**.

2. Let f be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Suppose that f' is strictly increasing and that $f(0) = 0$. Prove that $g(x) := f(x)/x$ is strictly increasing on $(0, \infty)$.

3. Suppose that f' exists and is continuous on $[0, 1]$. Suppose further than the series $\sum_{k=1}^{\infty} f(\frac{1}{k})$ converges. Prove that $f(0) = f'(0) = 0$.