Chapter 7: Integration

There are many different ways of defining integrals. Different definitions generally give the same values on the same functions; the difference is in which functions can be integrated and which cannot.

In 1821-1812, the Lebesgue integral is developed. This is very powerful but also very complex to define. In this course we will develop the Riemann integral. This is more powerful than the integral "defined" in most calculus courses, but much less powerful than the Lebesgue integral. The trade-off is that its definition is much quicker.

Definition

Fix $a < b$ in $\mathbb{R}$. A partition of $[a,b]$ is a finite set $P \subseteq [a,b]$ which contains $a$ and $b$. Typically we arrange its elements in order

$$a = x_0 < x_1 < \cdots < x_n = b$$

Given a partition $P$, write $\Delta_i = x_i - x_{i-1}$ ($1 \leq i \leq n$)

and let $\text{mesh}(P) = \max_{1 \leq i \leq n} \Delta_i$

Now let $f : [a,b] \to \mathbb{R}$ be bounded. Given a partition $P$ define

$$M_i(f, P) = \sup \{ f(x) \mid x_{i-1} \leq x \leq x_i \}$$

$$m_i(f, P) = \inf \{ f(x) \mid x_{i-1} \leq x \leq x_i \}$$

Finally, define the upper and lower Riemann sums

$$\mathcal{U}(f, P) = \sum_{i=1}^{n} M_i(f, P) \Delta_i$$ (upper)

$$\mathcal{L}(f, P) = \sum_{i=1}^{n} m_i(f, P) \Delta_i$$

Write $\mathcal{P}(a,b)$ be the set of all partitions of $[a,b]$. Given a bdd real function $f$ on $[a,b]$ define
\[ U(f) = \sup \{ U(f, P) : P \in \mathcal{P}(a, b) \} \quad \text{and} \quad L(f) = \inf \{ L(f, P) : P \in \mathcal{P}(a, b) \} \]

If \( U(f) = L(f) \), \( f \) is called Riemann integrable and the common value defines the Riemann integral
\[ \int_{a}^{b} f(x) \, dx. \]

**Lemma**

It follows from the boundedness of \( f \) that \( U_i \) and \( U_i \) exist, and so \( L(f, P) \) and \( U(f, P) \) exist. It is not perhaps immediately obvious \( U(f) \) and \( L(f) \) exist. But they do; as will follow from the following:

**Definition**

If \( P, Q \in \mathcal{P}(a, b) \) then \( Q \) is a refinement of \( P \), if \( Q \supset P \).

Given any \( P, Q \in \mathcal{P}[a, b] \) we can always find a common refinement of both, namely \( P \cup Q \).

**Lemma**

If \( R \) is a refinement of \( P \) (and \( f \) is \( f \))
\[ L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, P) \]

**Remark**

This justifies the use of \( \sup \) and \( \inf \) in the definition of \( U(f) \) and \( L(f) \) above. We're really looking for the numbers \( L(f, P) \) and \( U(f, P) \) approach as we take finer and finer partitions.

1. Clearly \( U(f, R) \leq U(f, L) \) and so \( L(f, L) \leq U(f, R) \)

2. Write \( P \) as \( a = x_0 < x_1 < \ldots < x_n = b \)

and consider \( x_{i-1}, x_i \) for a fixed \( i \)
Same \( L = \mathcal{P} \), \( x_i, x_{i-1} \in \mathbb{R} \), but there may be other pts of \( L \) between \( x_{i-1} \) and \( x_i \)

\[
\therefore x_{i-1} = t_k < t_{k+1} < \cdots < t_L = x_i
\]

Now for each \( k < j \leq l \) \( [t_{j-1}, t_j] \subseteq [x_{i-1}, x_i] \)

and so

\[
H_i(f, \mathcal{P}) = \sup_{x \in [t_{j-1}, t_j]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x) = H_i(f, \mathcal{P})
\]

and, likewise

\[
m_i(f, \mathcal{P}) = \inf_{x \in [t_{j-1}, t_j]} f(x) \geq \inf_{x \in [x_{i-1}, x_i]} f(x) = m_i(f, \mathcal{P})
\]

Thus

\[
\sum_{j=k+1}^{L} H_i(f, \mathcal{P}) (t_j - t_{j-1}) \leq \sum_{j=k+1}^{L} H_i(f, \mathcal{P}) (t_j - t_{j-1}) = H_i(f, \mathcal{P}) (x_i - x_{i-1})
\]

Summing from \( i = 1 \ldots n \) gives

\[ U(f, \mathcal{R}) \leq U(f, \mathcal{P}) \]

Likewise from (2), \( L(f, \mathcal{R}) \geq L(f, \mathcal{P}) \)

\[ \text{Corollary} \]

If, \( P, Q \in \mathcal{P}^a(b) \) then \( L(f, P) \leq U(f, Q) \)

if every lower sum is below every upper sum

Thus \( U(f, P) \) and \( L(f, Q) \) exist for any bounded \( f \).

If \( \mathcal{R} = P \cup Q \) then

\[ L(f, P) \leq L(f, \mathcal{R}) \leq U(f, \mathcal{R}) \leq U(f, Q) \].
Theorem 7.1 (Riemann's Criterion) Let $f$ be a bounded function on $[a,b].$

Then $f$ is Riemann integrable if

for any $\varepsilon > 0$ we can find a partition $P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Note: This is useful in the same way that Cauchy sequences are useful, or the comparison test. You can prove a function is integrable without knowing in advance what the integral is.

$p \Rightarrow$: Suppose $f$ is integrable. Let $\varepsilon > 0$ be given.

Then by def. of $U(f)$ as $\inf \{ U(f, P) : P \in P(a, b) \}$,

there is a partition $P_1$ so

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2}$$

Likewise, there is a partition $P_2$ so

$$L(f, P_2) > L(f) - \frac{\varepsilon}{2}$$

Thus, taking $P = P_1 \cup P_2$

$$U(f, P) - L(f, P) = U(f, P_1) - L(f, P_2) < U(f) - L(f) + \varepsilon$$

But since $f$ is integrable, $U(f) - L(f) = 0$.

$
\leq$ converses $U(f) - L(f)$. Clearly, since $U(f, P) = L(f, P)$ for all $P, Q$,

$U(f) - L(f) = 0$. On the other hand, given

any $\varepsilon > 0$, find a partition $P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Part $U(f) \leq U(f, P)$ and $L(f) \geq L(f, P)$ so

$$U(f) - L(f) = U(f, P) - L(f, P) < \varepsilon$$

Since $\varepsilon$ is arbitrary, $U(f) - L(f) \leq 0$. Thus $U(f) = L(f)$

and $f$ is integrable.
If \( f \) is increasing (resp. decreasing) on \([a, b]\) then it is Riemann integrable.

**Remark** Increasing functions need not be continuous or differentiable, so this is a large class of functions. On the other hand, clearly not all continuous functions are increasing. But are all continuous functions "built out of" an increasing and a decreasing part?

Test \( \sin(x) \) can be written as the difference of two increasing functions.

Can we do this for any other function? Answers later.
Let \( f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in \mathbb{R} \setminus A \end{cases} \) and fix \( a < b \).

Consider any partition \( a = x_0 < x_1 < \ldots < x_n = b \).

Then for any \( 1 \leq i \leq n \) \( [x_{i-1}, x_i] \) contains both rational and irrational \( p \) such that

\[
\max(f, p) = 0 \quad \text{and} \quad \min(f, p) = 1
\]

and so

\[
\max(f, p) = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i-1}) = b - a
\]

\[
\min(f, p) = 0
\]

\[
\max(f, p) - \min(f, p) = b - a > 0 \quad \text{for all } p
\]

Thus, by Riemann's condition, \( f \) is not Riemann integrable.

Goal

We now need to prove that all continuous functions are integrable.

To do this, we need a little diversion.

Recall \( f: S \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n \) is continuous on \( S \) if

\[
\forall x \in S \forall \epsilon > 0 \exists \delta > 0 \forall y \in S \quad \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon
\]

Definition: Let \( f: S \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n \). Say \( f \) is uniformly continuous on \( S \) if

\[
\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in S \quad \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon
\]

is true, then \( f \) is uniformly continuous on \( S \).

If the \( S \) you hand works for any pair \( x_i, y_i \), you don't have to fix \( x \) first.
Consider \( f(x) = x^2 \) which is uniformly \( \epsilon \)-continuous on \([a, b]\) but not on \( \mathbb{R}\).

1. Let \( S = [a, b] \) and note for \( x, y \in S \): 
   \[
   |f(x) - f(y)| = |x-y(x+y)| \\
   \leq (|x| + |y|) |x-y| \\
   \leq 2k |x-y| \quad \text{where} \quad k = \max |a|, |b|.
   
   So, given \( \epsilon > 0 \), take \( \delta = \epsilon / 2k + 1 \) and for any \( x, y \in S \) with \( |x-y| < \delta \), we have \( |f(x) - f(y)| < \epsilon \).

2. We need to show 
   
   \[
   \exists \epsilon_0 > 0 \quad \forall \delta > 0 \exists |x-y| < \delta \Leftrightarrow |x^2 - y^2| \geq \epsilon_0.
   
   In fact, taking \( \epsilon_0 = 1 \) works.

   Given any \( \delta \), take \( y = x + \frac{\delta}{2} \) so \( |x-y| < \delta \) and use 
   \[
   |x^2 - y^2| = |8x + \frac{\delta^2}{4}|.
   
   If we take \( x = \frac{1}{10} \) then \( |x^2 - y^2| = 1 + \frac{\delta^2}{4} \geq \epsilon_0 \).

   Rep. Every \( \epsilon \)-continuous function on a closed bounded interval \([a, b]\) is also \( \epsilon \)-continuous. 

Suppose for a contradiction that \( f \) is not uniformly \( \epsilon \)-continuous on \([a, b]\).

Then there is an \( \epsilon_0 > 0 \) and \( \forall \delta > 0 \exists |x-y| < \delta \) at 
\[
|f(x) - f(y)| > \epsilon_0.
   
   Take \( \delta_n = \frac{1}{n} \) and for each \( n \) find \( x_n, y_n \) such that 
   \[
   |x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| > \epsilon_0.
   
   But then for any \( n \), 
   
   \[
   |f(x_n) - f(y_n)| > \epsilon_0 \quad \text{for all} \quad n.
   
   Contradiction.
By the Bolzano–Weierstrass Theorem, $x_n$ has a convergent subsequence $x_{n_k} \to c$. Note that $a \leq x_n \leq b$ and so $a \leq c \leq b$.

Also, $|x_{n_k} - c| \leq |x_{n_k} - x_k| + |x_k - c|$

so by the Sandwich Theorem $y_{n_k} \to c$ too.

Now, $f \geq 0$ at $c$ and so there is a $\delta > 0$ so $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon_0/2$

Since $x_{n_k}$ and $y_{n_k}$ converge to $c$, there is an $N$ so $n > N \Rightarrow |x_{n_k} - c| < \delta$ and $|y_{n_k} - c| < \delta$

$\Rightarrow |f(x_{n_k}) - f(c)| < \varepsilon_0/2$ and $|f(y_{n_k}) - f(c)| < \varepsilon_0/2$

and so $|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(c)| + |f(c) - f(y_{n_k})| < \varepsilon_0$

canardining the construction.

The Picture of Uniform Continuity

$f \geq 0$ at $x_0$ means:

Given $\varepsilon > 0$, we can cut the tube of

at width $\delta > 0$ so that the curve exits through the sides.

Uniformly, it means:

Given $\varepsilon > 0$, find $\delta > 0$

so you can cut a tube of length $\delta$

d along the whole curve.
Theorem 7.4: Every $\epsilon$-close function on $[a,b]$ is Riemann integrable on $[a,b]$.

If $f$ is $\epsilon$-close on $[a,b]$ then it is uniformly $\epsilon$-close.

Let $\epsilon > 0$ be given. Find $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon' = \epsilon / (b-a)$.

Let $P$ be a partition with mesh less than $\delta$. Then for any $i$,

$$f(x_i) - f(x_{i-1}) < \epsilon' \quad \forall i$$

$$f(x) < f(x_{i-1}) + \epsilon' \quad \forall x \in [x_{i-1}, x_i]$$

$$\text{Hi}(P) < f(x_{i-1}) + \epsilon'$$

$$\text{Hi}(P) - \epsilon' < f(x_{i-1})$$

$$\text{Hi}(P) - \epsilon' < \text{Hi}(P)$$

$$\text{Hi}(P) - \text{Mi}(P) < \epsilon'$$

$$\text{Hi}(P) - \text{Mi}(P) = \sum_{i=1}^{n} (\text{Hi}(P_i) - \text{Mi}(P_i)) (x_i - x_{i-1})$$

$$< \epsilon' \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \epsilon' (b-a) = \epsilon.$$

Thus, for $\delta$, $f$ is integrable.

Proposition 7.5: Suppose $f, g$ are Riemann integrable on $[a,b]$ and $c, d \in \mathbb{R}$ then

$$(c f + d g) \text{ is Riemann integrable and}$$

$$\int_{a}^{b} (cf(x) + dg(x)) \, dx = c \int_{a}^{b} f(x) \, dx + d \int_{a}^{b} g(x) \, dx$$

Note it suffices to prove the result for $c=d=1$ and for $f = 0$.

(i.e., $f + g$ and $cf$)

Case 1: $f + g$. Clearly for any $P$, $\text{Hi}(f + g)$ is an upper bound for $f + g$ and $\text{Mi}(f + g)$ is a lower bound.
Thus
\[ \min(f(x), p) + \max(g(x), p) \leq \min(f + g, p) \leq \max(f + g, p) \]

**1.**
\[ L(f(x), p) + L(g(x), p) \leq L(\min(f + g, p)) \leq \min(L(f, p), L(g, p)) \]

Now consider two partitions \( P, Q \in \mathcal{P}(a, b) \) and let \( E = P \cup Q \) so that

2a.
\[ \mathcal{U}(f + g, E) = \mathcal{U}(f, E) + \mathcal{U}(g, E) \leq \mathcal{U}(f, P) + \mathcal{U}(g, Q) \]

and likewise

2b.
\[ \mathcal{L}(f + g, E) = \mathcal{L}(f + g, P) = \mathcal{L}(f, E) + \mathcal{L}(g, E) \geq \mathcal{L}(f, P) + \mathcal{L}(g, Q) \]

Now take \( \inf \) of (2a) as \( P, Q \) range across \( \mathcal{P}(a, b) \) so that
\[ \mathcal{U}(f + g) = \mathcal{U}(f) + \mathcal{U}(g) = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

Likewise take \( \sup \) of (2b) as \( P, Q \) range across \( \mathcal{P}(a, b) \) so that
\[ \mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g) = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

Thus, a priori, \( \mathcal{U}(f + g) \geq \mathcal{L}(f + g) \)

\[ \mathcal{U}(f + g) - \mathcal{L}(f + g) = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

Next, suppose \( f \) is integrable and \( \alpha > 0 \)

Given \( P, \) \[ \alpha \mathcal{U}(f, P) \leq \alpha f(x) \leq \alpha \mathcal{L}(f, P) \] on \([x_i, x_{i+1}]\)

1. \[ \alpha \mathcal{U}(f, P) \leq \mathcal{U}(\alpha f, P) \leq \mathcal{U}(\alpha f, P) \leq \alpha \mathcal{U}(f, P) \]

2. \[ \alpha \mathcal{L}(f, P) \leq \mathcal{L}(\alpha f, P) \leq \mathcal{L}(\alpha f, P) \leq \alpha \mathcal{L}(f, P) \]

3. \[ \alpha \int_a^b f(x) \, dx = \alpha \mathcal{L}(f) \leq \mathcal{L}(\alpha f) \leq \mathcal{U}(\alpha f) \leq \alpha \int_a^b f(x) \, dx \]

Finally,

\[ -\mathcal{U}(f, P) \leq -f(x) \leq -\mathcal{L}(f, P) \]

\[ -\mathcal{U}(f, P) \leq \min(-f, P) \leq \max(-f, P) \]
Thus if \( \alpha < 0 \), \( \alpha f = -|\alpha| f \) is integrable and
\[
\int_a^b \alpha f(x) \, dx = -|\alpha| \int_a^b f(x) \, dx = -|\alpha| (b-a) f(a) = \alpha \int_a^b f(x) \, dx.
\]

Prop. 7.6. Suppose \( f \) is Riemann integrable. Then \( |f(x)| \) is also Riemann integrable and
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.
\]

Recall "reverse triangle inequality" \( |a| - |b| \leq |a - b| \). Let \( P \in \Pi(a, b) \), then for \( x, y \in [x_i, x_{i+1}] \)
\[
|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)|.
\]

Add \( y \) constant and take \( \sup \) over \( x \)
\[
\sup \left| \int_{x_i}^{x_{i+1}} f(x) \, dx \right| \leq \sup \int_{x_i}^{x_{i+1}} |f(x)| \, dx = \sup \| f(x) - f(x_i) \| = \| f \| - f(x_i).
\]

Now take \( \inf \) over \( y \)
\[
\inf \left| \int_{x_i}^{x_{i+1}} f(x) \, dx \right| \leq \inf \int_{x_i}^{x_{i+1}} |f(x)| \, dx = \inf \| f(x) - f(x_i) \| = \| f \| - f(x_i).
\]

So, by Riemann's criterion, \( f \) is Riemann integrable.

The inequality now follows from the lemma, and
\[
|f| \leq f \leq |g|.
\]

Lemma. If \( f, g \) are Riemann integrable and \( f \leq g \) then
\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]

Clearly
\[
\| f \| - f(x_i) \leq \| g \| - g(x_i) \quad \text{and} \quad f(x_i) - f(x) \leq g(x_i) - g(x).
\]
Theorem 7.7 (Fundamental Theorem of Calculus - I) Let $f$ be Riemann integrable on $[a, b]$ and suppose there is a continuous function $F$ on $[a, b]$ such that:

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

Then:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Let $\epsilon > 0$ and find $P_1$ such that:

$$U(f, P_1) < U(f, P_2) + \epsilon$$

and $P_2$ such that:

$$L(f, P_2) > L(f) - \epsilon$$

Taking $P = P_1 \cup P_2$,

$$U(f, P) = U(f, P_1) < U(f, P_2) + \epsilon \quad \text{and} \quad L(f, P) = L(f, P_2) > L(f) - \epsilon$$

Now consider $[x_{i-1}, x_i]$ from $P$. Since $F$ is continuous on $[x_{i-1}, x_i]$ and differentiable on $(x_{i-1}, x_i)$,

$$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$$

where $c_i \in (x_{i-1}, x_i)$ and so

$$m_i(f, P_i) \leq f(c_i) \leq M_i(f, P_i)$$

Thus:

$$F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

$$= \sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1})$$

$$\Rightarrow$$

$$F(b) - F(a) \leq \sum_{i=1}^{n} M_i(f, P_i)(x_i - x_{i-1}) = U(f, P) < U(f) + \epsilon$$

and

$$F(b) - F(a) \geq \sum_{i=1}^{n} m_i(f, P_i)(x_i - x_{i-1}) = L(f, P) > L(f) - \epsilon$$

But $U(f) - L(f) = \int_a^b f(x) \, dx$ and so

$$\left| \left( F(b) - F(a) \right) - \int_a^b f(x) \, dx \right| < \epsilon$$

Since the lower bound is independent of $\epsilon$. 

Therefore, the integral exists.
Lemma

Let $f$ be Riemann integrable on $[a, b]$ and let $a < c < b$. Then $f$ is Riemann integrable on $[a, c]$ and on $[c, b]$, and

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

See Analytic Webnotes

Then 7.8 (Fund Thm Calc II) Let $f$ be Riemann integrable on $[a, b]$. For $x \in [a, b]$

Define

$$F(x) = \int_a^x f(t) \, dt$$

Then $F$ is cts on $[a, b]$. Furthermore, if $f$ is cts at $c \in [a, b]$, then $F$ is diffble at $c$ (left/right limits at end pts) and

$$F'(c) = f(c)$$

Proof

Since $f$ is Riemann integrable on $[a, b]$, it is bounded, say by $M$.

Suppose $x > x_0$, then

$$|F(x) - F(x_0)| = \left| \int_{x_0}^x f(t) \, dt - \int_{x_0}^{x_0} f(t) \, dt \right|$$

$$= \left| \int_{x_0}^x f(t) \, dt \right|$$

by lemma

$$\leq \int_{x_0}^x |f(t)| \, dt$$

by triangle inequality

$$\leq \int_{x_0}^x M \, dt$$

by lemma

$$= M (x - x_0)$$

by FTC I, or 1st principles

$$= M |x - x_0|$$

likewise, if $x < c$, verify

$$|F(x) - F(x)| \leq M |x - x_0|$$

and continuity follows.

@ Likewise, suppose $f$ is cts at $c$ and $x > c$

$$F(x) - F(c) = \int_c^x f(t) \, dt$$

and

$$f(c)(x - c) = \int_c^x f(c) \, dt$$
Thus
\[ \int_c^x f(t) - f(c) \, dt = (F(x) - F(c)) - f(c) (x - c) \]

Thus
\[ \frac{F(x) - F(c)}{x - c} = \frac{1}{x - c} \int_c^x f(t) - f(c) \, dt \]  

By a similar calculation, the result holds for \( x < c \) too.

Now, again \( \epsilon > 0 \), find \( \delta > 0 \) such that \( |x - c| < \delta \) (and \( x \in [a, b] \))
\[ |f(x) - f(c)| < \epsilon \]
Thus, if \( x \in [a, b] \) and \( |x - c| < \delta \),
\[ \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)| \, dt \]
\[ \leq \frac{1}{|x - c|} \int_c^x \epsilon \, dt \]
\[ = \frac{\epsilon}{|x - c|} \cdot |x - c| = \epsilon. \]

Thus \( F \) is differentiable at \( c \) and \( F'(c) = f(c) \).

**Improper Integrals**

**Def.** Let \( f \) be defined on \([a, +\infty)\) and suppose \( f \in \mathbb{R}(a, b) \) for every \( b > a \). Then \( \int_a^b f(t) \, dt \) exists for every \( b > a \). If

further, \( L = \lim_{b \to +\infty} \int_a^b f(t) \, dt \) exists, then the improper integral
\[ \int_a^\infty f(t) \, dt \]
exists, with value \( L \).
Def. Define $\int_a^b f(x)\,dx$ analogously. Finally if $f \in R(a,b)$ for all $a \leq b$ in $\mathbb{R}$, define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

if both exist.

Note: $\int_a^b f(x)\,dx$ does not depend on $a$. (Hole)

2. $\int_{-L}^L f(x)\,dx = 0 \Rightarrow \lim_{L \to \infty} \int_{-L}^L f(x)\,dx = 0$ defines $\int_{-\infty}^{\infty} f(x)\,dx = 0$

if any value is possible.

Prop 7.9 (Integral Cauchy Test) Let $f(x)$ be nonnegative, decreasing on $[1, \infty)$. Then $\sum_{n=1}^{\infty} f(n)$ converges if $\int_1^{\infty} f(x)\,dx$ converges.

Since $f$ is monotonous on $[1, b]$, let $Q_b(1, b)$ for all $b > 1$.

Since $f(x) \geq 0$, $\int_1^{b} f(t)\,dt$ is increasing in $b$ and thus $\lim_{b \to \infty} \int_1^{b} f(t)\,dt$ exists if $\int_1^{\infty} f(t)\,dt$ is bounded above.

Likewise the partial sums of $\sum_{n=1}^{\infty} f(k)$ are increasing and so the series converges if the partial sums are bounded above.

Now let $n \in \mathbb{N}$ and consider the partition $x_i = i + 1$ for $i = 0, 1, 2, \ldots, n$ of $[1, n+1]$. Clearly $f(i+1) \leq f(x) \leq f(i)$ for $x \in [x_{i-1}, x_i]$. Then

$$\min f(P) = f(i+1), \quad \max f(P) = f(i)$$

and

$$L(f, P) = \sum_{i=1}^{n} f(i+1) = \sum_{k=2}^{n+1} f(k)$$
and \( W(f, p) = \sum_{i=1}^{n} f(i) \).

Thus,
\[
\int_{1}^{n+1} f(x) \, dx = W(f) = W(f, p) = \sum_{i=1}^{n} f(i).
\]

So if \( \sum_{k=1}^{n} f(k) \) converges then \( \int_{1}^{n+1} f(x) \, dx \) is bounded and increasing.

Conversely,
\[
\int_{1}^{n+1} f(x) \, dx = W(f) = W(f, p) = \sum_{k=1}^{n+1} f(k)
\]

and so if
\[
\int_{1}^{\infty} f(x) \, dx \text{ converges, the partial sums } \sum_{k=1}^{\infty} f(k) \text{ are bounded.}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}
\]

\[
\int_{e}^{L} \frac{1}{x} \, dx = \int_{1}^{\ln L} \frac{1}{u} \, du \quad \text{where } u = \ln x, \quad du = \frac{1}{x} \, dx
\]

\[
= \ln(\ln L) \quad \text{as } L \to \infty.
\]

**Def.** If \( f \) is defined on \((a, b]\) and \( f \in \Theta(a, b) \) for all \( a' \in (a, b) \)
then say the improper integral
\[
\int_{a}^{b} f(x) \, dx
\]
exists and is equal to \( L \) if
\[
\lim_{a' \to a^+} \int_{a'}^{b} f(x) \, dx = L
\]
\[ \int_0^1 \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \int_t^1 x^{-\frac{1}{2}} \, dx = \lim_{t \to 0^+} 2(1 - \sqrt{t}) = 2 \]

Riemann–Stieltjes Integration

There are a lot of applications where one wants to work with an integral of the form
\[ \int_a^b f(x) \, g'(x) \, dx \]

Let \( X \) be a random variable, \( g(x) = P(X \leq x) \) is the CDF function, \( g'(x) \) is the PDF function.

\[ E(X) = \int x \, g'(x) \, dx \]

But what if \( g' \) doesn't exist, say \( g \) has some discontinuities?

Can we formally write
\[ \int_a^b f(x) \, g'(x) \, dx = \int_a^b f(x) \, d\,g(x) \]

So if \( \int_a^b f(x) \, dx \) be approx by \( \sum \frac{1}{n} f(c_i) (x_i - x_{i-1}) \)
then \( \int_a^b f(x) \, d\,g(x) \) should be approx by \( \sum \frac{1}{n} f(c_i) (g(x_i) - g(x_{i-1})) \)

The integral we get this way is called the Riemann–Stieltjes integral

*One big difference is \( g(x_i) - g(x_{i-1}) \) need not to positive!
Preview of Cool Facts

\[ \int_a^b f(x) \, dx = \int_a^b f(x) \, g(x) \, dx \quad \text{if} \quad g(x) \text{ is a } C^1 \text{ function} \]

**Dear delta**

\[ \int_a^b f(x) \, dx = f(c) \quad \text{if} \quad g(x) = 0 \quad \text{for} \quad x < c \quad \text{and} \quad f \text{ is continuous at } c \]

**Substitution**

\[ \int_a^b f(g(x)) \, g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du \quad \text{under suitable conditions} \]

**but try Facts**

\[ \int_a^b f(x) \, dx = \int_a^b f(x) \, g(x) \, dx + \int_a^b g(x) \, df(x) = f(b)g(b) - f(a)g(a) \]

**Def.** Let \( f, g \) be bounded on \([a, b]\). Given a partition \( P = \{x_0, \ldots, x_n\} \) of \([a, b]\), call \( \{\xi_1, \ldots, \xi_n\} \) an evaluation sequence for \( P \) if each \( \xi \in [x_{i-1}, x_i] \).

**Given a partition \( P \) and an evaluation sequence \( \xi \),

\[ I_{g\xi}(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i) \left( x_i - x_{i-1} \right) \]

the Riemann-Stieltjes sum \( \langle f, g, P, \xi \rangle \)

**Def.** Say that \( f \in R(g) \), or \( f \) is Riemann-Stieltjes integrable w.r.t \( g \), if there exists an \( L \in \mathbb{R} \) such that for any \( \varepsilon > 0 \) we can find a partition \( P \) of \([a, b]\) and an evaluation sequence \( \xi \) such that for all \( P \geq P_\varepsilon \) and all \( \xi \),

\[ \left| I_{g\xi}(f, P, \xi) - L \right| < \varepsilon \]

It is clear that \( \int_a^b f(x) \, dx = L \)

**Note**: \( f \) is the integrand; \( g \) is the integrator.
let \( g(x) = x \) Then \( f \in \mathbb{Q}(a) \) iff \( f \in \mathbb{Q}(a, b) \) and
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx
\]

Homework Hint

\[
L(f, P) \leq I_\delta(f, P, \varepsilon) \leq U(f, P)
\]

Let \( g(x) = \begin{cases} 1 & x > c \\ 0 & x \leq c \end{cases} \) where \( c \in (a, b) \) is fixed.

Suppose \( \lim_{x \to c^+} f(x) = f(c) \) Then \( f \in \mathbb{Q}(a) \) and \( \int_a^c f(x) \, dx g(x) = f(c) \)

Let \( \varepsilon > 0 \) be given

\[
\text{Since } \lim_{x \to c^+} f(x) = f(c) \text{, find } \delta > 0 \text{ s.t. } 0 < x - c < \delta \implies |f(x) - f(c)| < \varepsilon.
\]

Let \( P_c \) consist of \( a < c < d < b \)

where we have fixed \( d \) between \( c \) and \( a \) with \( d < c + \delta \)

Now let \( P = P_c \) be a partition and \( \varepsilon \) an error sequence for it

Since \( P \supseteq P_c \), there is an \( i_0 \) s.t. \( x_{i_0} = c \)

Now for \( i < i_0 \), \( g(x_i) - g(x_{i-1}) = 0 - 0 = 0 \)

\[
\text{and } g(x_i) - g(x_{i-1}) = 1 - 1 = 0
\]

Thus

\[
I_{\delta}(f, P, \varepsilon) = f(x_i)
\]

Hence \( x = c \leq x_i \leq x_i < d < c + \delta \)

and so

\[
|I_{\delta}(f, P, \varepsilon) - f(c)| = |f(x_i) - f(c)| < \varepsilon.
\]
Lemma 7.10 Let \( a(x) = x \). Then \( f \in R(a) \) iff \( f \in R(a,b) \) and the integrals
\[
\int_a^b f(x) \, da(x) \quad \text{and} \quad \int_a^b f(x) \, dx
\]
agree.

Example: If \( P \) is a partition and \( \{x_i \} \) an eval sequ
\[
\text{L}(f, P) \leq \sum f(x_i) (x_i - x_{i-1}) \leq \text{U}(f, P)
\]
\[
= \sum f(x_i) (a(x_i) - a(x_{i-1}))
\]

Lemma 7.11 If \( f, f_2 \in R(a) \) and \( a_1, a_2 \in E \) then \( a_1 + x_2, b \in R(a) \) and
\[
\int_a^{b} [f_1(x) + f_2(x)] \, dx = \int_a^{b} f_1(x) \, dx + \int_a^{b} f_2(x) \, dx
\]
(1) If \( f, f_2 \in R(a) \) and \( f_1 \in R(b) \) and \( a, x \in E \) then
\[
f \in R(a, b)
\]
and
\[
\int_a^{b} f(x) \, dx + \int_a^{b} f_2(x) \, dx = \int_a^{b} f(x) \, dx + \int_a^{b} f_2(x) \, dx
\]
(2) If \( f \) is L-S integrable on \( [a, b] \) and also \( \phi \) on \([c, d] \)
then \( f + \phi \) is L-S integrable on \([a, b] \)
and
\[
\int_a^{b} f(x) \, dx \phi(x) = \int_a^{c} f(x) \, dx \phi(x) + \int_c^{b} f(x) \, dx \phi(x)
\]

Proof (1)

(2) Write \( L_i = \int_a^{b} f \, dx \) and find \( P_i \) st \( f \in R(P_i) \) for all \( P \supset P_i \) and eval sequ \( \{x_i \} \)
\[
| I_i(f, P, \varepsilon) - L_i | < \frac{\varepsilon}{\sqrt{1 + \sqrt{1 + 1}}}
\]

Let \( P = P_i \cup P_2 \) and let \( P \supset P_i \), \( \varepsilon \) an eval sequ
\[
I_{x_i + \varepsilon} (f, P, \varepsilon) = \sum f(x_i) \left( (x_i + \varepsilon) - x_i \right) - \left( x_{i+1} - (x_i + \varepsilon) \right)
\]
\[
= \alpha_1 I_{x_i + \varepsilon} (f, P, \varepsilon) + \alpha_2 I_{x_i} (f, P, \varepsilon)
\]
Let \( f, g \) be bounded on \([a, b]\). Suppose \( \phi : [c, d] \to [a, b] \) is an increasing bijection. If \( f \in L^1(\phi) \) then \( f \circ \phi \in L^1(\phi^{-1}) \)
and
\[
\int_c^d f \circ \phi \, dx = \int_a^b f \circ \phi \circ \phi^{-1} \, dx.
\]

Let \( L = \int_a^b f \, dx \) and find \( P_0 = P \) for all \( P \in P_0 \) and \( \epsilon > 0 \).

Since \( \phi \) is a bijection, let \( P_1 = \{ \phi^{-1}(x_i) \mid i = 1, \ldots, n \} \)
and since \( \phi \) is increasing,
\[
c = \phi(a) < \phi^{-1}(x_1) < \cdots < \phi^{-1}(x_n) = d
\]
then \( P_1 \) is a partition of \([c, d]\).

Now if \( P \neq P_1 \) and \( t \in \text{eval} \), \( \text{eval} \),
write \( P = \{ c = w_0 < \cdots < w_k = d \} \) for \( t = \sum w_i \).

So
\[
I_{g \circ \phi}(f \circ \phi, P, t) = \sum_i f(\phi(w_i)) (g(\phi(w_i)) - g(\phi(w_{i-1})))
\]
where
\[
P' = \{ a = \phi(c), \phi(c), \cdots, \phi(d) = b \}
\]
and \( t' = \sum \phi(w_i) \) where \( \phi(w_i) \leq \phi(w_{i-1}) \).

So \( P' \) is a partition of \([a, b]\) and \( t' \) is an \( \text{eval} \), \( \text{eval} \), \( \text{eval} \), \( \text{eval} \), \text{eval} \).

Thus \( P \neq P_0 \) so \( P' \neq P_0 \)

\[
\left| I_{g \circ \phi}(f \circ \phi, P, t) - L \right| < \epsilon
\]
Suppose \( f \in \mathbb{R}(a) \). Then, as \( a \in (f) \) and
\[
\int_a^b f(x) \, dg(x) + \int_a^b g(x) \, df(x) = f(b) \, g(b) - f(a) \, g(a)
\]

Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\) and \((x_i)^n\) an evaluation sequence.

Of course,
\[
I(f, P, x) = \sum_{i=0}^{n-1} g(x_i) \left( f(x_i) - f(x_{i-1}) \right) = \sum_{i=1}^n f(x_i) g(x_i) - f(x_{i-1}) g(x_i)
\]
and
\[
f(b) g(b) - f(a) g(a) = \sum_{i=1}^n f(x_i) g(x_i) - f(x_{i-1}) g(x_{i-1})
\]
and so
\[
-I(f, P, x) + f(b) g(b) - f(a) g(a) = \sum_{i=1}^n f(x_i) (g(x_i) - g(x_{i-1})) + \sum_{i=1}^n f(x_{i-1}) (g(x_i) - g(x_{i-1}))
\]
\[
= I(f, P_{U^2}, W)
\]
where \( W \) is a suitable evaluation sequence for \( P_{U^2} \).

In principle, \( W \) assigns \( x_i \) to \([x_i, x_{i+1}]\)
and \( x_{i-1} \) to \([x_{i-1}, x_i]\)

with the caveat that we must ensure where \( x_i = x_{i-1} \) or \( x_i = x_{i+1} \)

Now, let \( \varepsilon > 0 \) be given. Since \( f \in \mathbb{R}(a) \), find \( P_0 \) so
for all \( P \geq P_0 \) and eval seq \( x \)
\[
I_g(f, P, x) - \int_a^b f(x) \, dg(x) \mid < \varepsilon
\]
Now if \( P \geq P_0 \) and \( x \) is an eval seq \( w + \varepsilon \) of course \( P_{U^2} \geq P \geq P_0 \) and so
\[
I_g(f, P_{U^2}, W) - \int_a^b f(x) \, dg(x) \mid < \varepsilon
\]
Thus儿 for any \( P \geq P_0 \) and \( x \)
\[
\left| I_g(f, P, x) - \left( f(b) g(b) - f(a) g(a) - \int_a^b f(x) \, dg(x) \right) \right| = \left| -I_g(f, P, x) + f(b) g(b) - f(a) g(a) - \int_a^b f(x) \, dg(x) \right| < \varepsilon.
\]
Suppose \( f \in Q(a, b) \). If \( g \in C^1 \) then \( f'g \in Q(a, b) \) and
\[
\int_a^b f(x)g(x) \, dx = \int_a^b f(x)g'(x) \, dx.
\]

Recall (Prop 7.10) that a function is Riemann integrable if it is RS integrable with the Riemann integral \( L(x) = x \) and the R- and L-rules coincide. Thus we shall show \( f(x)g'(x) \) is RS integrable with \( L(x) = x \) and that its integral is \( \int_a^b f(x)g(x) \, dx \).

Now we apply Prop 7.11. The Mean Value Theorem gives \( C_e(x, a, b) \) such that
\[
g(x) - g(c) = g'(c)(x - c),
\]
and so
\[
I_g(f, P, \tau) = \sum f(c_i)g'(c_i)(x_i - c_i).
\]
This is not quite the same as
\[
I_L(f, P, \tau) = \sum f(c_i)g'(c_i)(x_i - c_i),
\]
but we can ascertain that they are close as follows:

Since \( g' \) is continuous on \([a, b]\), it is uniformly continuous. Let \( \epsilon > 0 \) be given and find \( \delta > 0 \) such that
\[
|x - y| < \delta \implies |g(x) - g(y)| < \epsilon.
\]
Now let \( P \) be a partition with mesh \( \xi \) less than \( \delta \). If \( P \geq P_0 \)
\[
|I_g(f, P, \tau) - I_L(f, P, \tau)| \\leq \sum |f(c_i)| |g'(c_i) - g'(\xi_i)| (x_i - c_i) \\
< M \epsilon (b - a) \text{ where } M = \sup |f(x)|.
\]
Now, since \( f \in Q(a, b) \), find \( P_1 \) such that \( P \geq P_1 \) and
\[
|I_g(f, P_1, \tau) - \int_a^b f(x)g(x) \, dx| < \epsilon.
\]
So if \( P \geq P_2 = P_0 \cup P_1 \) then
\[
|I_L(f, P, \tau) - \int_a^b f(x)g(x) \, dx| < (M(b - a) + 1) \epsilon.
\]
Consider $\int_{-1}^{1} e^{x} dH(x)$ where $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$.

From a previous example we know $\int_{-1}^{1} e^{x} dH(x) = e^0 = 1$. Now we compute

$$\int_{-1}^{1} e^{x} dH(x) = e^{H(1)} - e^{H(-1)} - \int_{-1}^{1} H(x) e^{x} dx$$

$$= e - \int_{-1}^{1} H(x) e^{x} dx = e - (e - 1) = 1$$

Similarly,

$$\int_{-1}^{1} H(x) e^{x} dx = \int_{0}^{1} H(x) e^{x} dx + \int_{-1}^{0} H(x) e^{x} dx$$

Now on $[0, 1]$ let $H(x) e^{x} - e^{x} = \begin{cases} 1 & x = 0 \\ 0 & x > 1 \end{cases}$

and for $x > 0$, and $e > 0$

$$\left| \int_{0}^{e} a(x) dx \right| \leq \left| \int_{0}^{e} a(x) dx \right| + \left| \int_{e}^{\infty} a(x) dx \right| = e$$

let $a(x) = \begin{cases} x+1 & x > 0 \\ 0 & x \leq 1 \end{cases}$

Calculate $\int_{-1}^{1} x^2 a(x) dx$

$\int_{-1}^{1} x^2 a(x) H(x) = 2 - \int_{-1}^{1} x(x+1) H(x) dx^2$

$$= 2 - \int_{-1}^{1} 2x(x+1) H(x) dx$$

$$= 2 - \int_{-1}^{1} 2x^2 + 2x dx$$

$$= 2 - \left( \frac{2}{3} + 1 \right) = 2 \frac{1}{3}$$
Let \( f \) be \( d \mathbb{R} \) on \( \mathbb{R} \) then \( \int_{-1}^{\infty} f(x) \, dH(x) = f(0) \).

Consider \( E(x) = e^{-x^2} \). Recall \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \).

Let \( g_a(x) = \frac{1}{\sqrt{a}} \, e^{-\frac{x^2}{a}} \) and so \( \int_{-\infty}^{\infty} g_a(x) \, dx = 1 \).

Now let \( f \) be \( d \mathbb{R} \) on \([-1, 1]\) and let \( \epsilon > 0 \) be given. Find a \( \delta > 0 \) such that \( |x| < \delta \Rightarrow |f(x) - f(0)| < \epsilon \).

Find \( N > \frac{1}{\delta} \) so that if \( n \in \mathbb{N} \) and \( |x| = \delta \), then
\[ \frac{\epsilon - |nx|^2}{|x|^2} < \epsilon \]

\( \frac{\epsilon - |nx|^2}{|x|^2} < \epsilon \)

\[ |\int_{-\infty}^{\infty} f(x) \, g_a(x) \, dx - f(0)| = |\int_{-\infty}^{\infty} (f(x) - f(0)) \, g_a(x) \, dx| \]
\[ \leq \int_{-\infty}^{\infty} |f(x) - f(0)| \, g_a(x) \, dx \]
\[ = \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty} \]
So was \( \int_{-b}^{b} |f(x) - f(0)| \, g_n(x) \, dx \leq \varepsilon \int_{-b}^{b} g_n(x) \, dx \)

\[ \leq \varepsilon \int_{-\infty}^{\infty} g_n(x) \, dx = \varepsilon. \]

\( \int_{-b}^{b} |f(x) - f(0)| \, g_n(x) \, dx \)

\[ \leq 2M \int_{-b}^{b} g_n(x) \, dx \]

\[ = \frac{2M}{b^n} \int_{-b}^{b} e^{-nx^2} \, dx \]

\[ \leq \frac{2M}{b^n} \int_{-\infty}^{\infty} e^{-nx^2} \, dx \]

\[ = \frac{2M}{b^n} \left( e^{-n\delta^2} - e^{-n} \right) \xrightarrow{\delta \to 0} b \text{ as } n \to \infty \]

Likewise, \( \int_{-b}^{b} |f(x) - f(0)| \, g_n(x) \, dx \xrightarrow{\delta \to 0} 0 \text{ as } n \to \infty \)

So there is an \( N_2 \) so \( n \geq N_2 \Rightarrow \int_{-b}^{b} |f(x) - f(0)| \, g_n(x) \, dx < 3\varepsilon \) and so

\[ \int_{-b}^{b} |f(x) - f(0)| \, g_n(x) \, dx < 3\varepsilon. \]

In all of these examples, the following version of the triangle inequality was essential:

if \( |f(x)| \leq M \) on \([a,b] \) then

\[ \left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx \leq M (b-a) \]

So what is the version of this for \( f \in L^1([a,b]) \)? Suppose \( |f(x)| \leq M \)

\[ \left| \int_{a}^{b} f(x) \, dg(x) \right| \]

is close to

\[ \leq \sum_{i=1}^{n} |f(x_i)| |g(x_i) - g(x_{i-1})| \]

\[ \leq M \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \]
However, when \( g \) is not increasing, \( \sum_{i=1}^{n} [a(x_i)-a(x_{i-1})] \) is not a telescoping sum.

**Def.** Let \( a: [a,b] \rightarrow \mathbb{R} \) and \( P \) a partition of \([a,b] \). The variation of \( a \) over \( P \) is
\[
V(a, P) := \sum_{i=1}^{n} [a(x_i)-a(x_{i-1})]
\]

**Lemma.** If \( P \subset Q \) then \( V(f, P) \leq V(f, Q) \)

\[
\text{Let } P \subset P \text{ where } P \text{ has exactly one point, } c, \text{ that is not in } P.
\]

So \( \forall \, x_{i-1} < c < x_i \) for some \( i \)

\[
|f(x_i) - f(x_{i-1})| = |f(x_i) - f(c)| + |f(c) - f(x_{i-1})|
\]

Since all the other terms of \( V(f, P) \) and \( V(f, P_i) \) are the same,
\[
V(f, P) \leq V(f, P_i)
\]

But now can construct
\[
P = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_k = Q
\]

where \( P_i \setminus P_{i-1} \) is a singleton.

**Def.** If \( \{V(f, P) : P \in \mathcal{P}(a,b)\} \) is bounded, say \( V_b \) is of bounded variation on \([a,b]\). Define the total variation as
\[
V_b f := \sup \{ V(f, P) : P \in \mathcal{P}(a,b) \}
\]

**Theorem 7.15.** \( f \) is of bounded variation if there exist increasing functions \( f_1, f_2 \) s.t. \( f = f_1 - f_2 \)

**Theorem 7.16.** Suppose \( |f(x)| \leq M \) on \([a,b]\), \( a \in Bv(a,b) \) and \( f \in Q(a,b) \). Then
\[
\left| \int_{a}^{b} f(x) \, dx \right| \leq M V_b a
\]
\[ f(x) = \begin{cases} \frac{x \cos \frac{1}{x}}{2} & x > 0 \\ 0 & x = 0 \end{cases} \]

\( p = \text{cts on } [0,1] \text{ but not } BV \)

\( \{ x_k \} \text{ covers } \frac{1}{k\pi} \text{ and } p = \langle 0, x_{(0)}, x_{(1)}, \ldots, x_{(n)}, 1 \rangle \}

Then:

\[ \| f_p \|_p = \sum_{k=2}^{N} \frac{1}{k\pi} \left| f(x_k) - f(x_{k-1}) \right| \]

\[ = \sum_{k=2}^{N} \left| \frac{1}{k\pi} \left( \frac{1}{k} - (k-1) \right) \right| \]

\[ = \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} \]

Thus, \( \{ |f_p| \} \) \( p = \text{not } BV \text{ above.} \)

\[ f = \text{cts on } [a,b], \text{ differentiable on } (a,b), \text{ and } f' = \text{bd} \text{ then } f \text{ BV} \]

\[ \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{i=1}^{n} \left| f'(c_i) \right| \left| x_i - x_{i-1} \right| \]

\[ \leq B \sum_{i=1}^{n} x_i - x_{i-1} = B(b-a) \]

Thus, any \( \{ x \cos \frac{1}{x} \} \text{ is } BV \text{ on } [0,1] \)
Before proving 7.16 we have some lemmas

**Lemma 7.16.** If $f, g$ are BV then $f + g$ is BV and

$$V_a^b f + V_a^b g \leq |b-a| \cdot V_a^b f + |b-a| \cdot V_a^b g$$

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**Homework**

**Lemma 7.16.** If $a < b < c$ then

$$V_a^c f \leq V_a^b f + V_b^c f$$

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Davidson-Doevariant notes

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**Proof of 7.16.**

(E1) Suppose $g$ is increasing. For any $P$

$$V(a, P) = \sum |g(x_i^+) - g(x_i^-)| = \sum (g(x_i^+) - g(x_i^-)) = g(b) - g(a)$$

So $V_a^b g = g(b) - g(a) < \infty$.

So if $f = f_1 - f_2$ where $f_1, f_2$ are increasing

$$V_a^b f \leq V_a^b f_1 + V_a^b f_2 < \infty$$

(9) Suppose $f$ is BV on $[a, b]$. Fix $x$ in $[a, b]$ and define

$$f_1(x) = V_x^b f$$

Now for any $y < x$

$$f_1(y) = V_y^b f - V_x^b f + V_x^b f = f_1(x) + V_x^b f \geq f_1(x)$$

so $f_1$ is increasing and $f_1(x) \leq f_1(b) = V_a^b f < \infty$ so $f_1(x)$ is always finite. So define

$$f_2(x) = f_1(x) - f(x)$$

Clearly $f = f_1 - f_2$ so it is enough to show $f_2$ increasing.

Put $y < x$

$$f_2(y) - f_2(x) = f_1(y) - f_1(x) - (f(x) - f(y)) = V_x^b f - (f(x) - f(y))$$

$$= V_x^b f - |f(x) - f(y)| \geq 0$$
Let $\epsilon > 0$ and find $P_0 \in \mathcal{P}$.

$$\left| \int_a^b f(x) \, dx - I_{P_0} \right| < \epsilon$$

$$\left| \int_a^b f(x) \, dx - I_{P_0} \right| < \epsilon$$

$$\left| \int_a^b f(x) \, dx \right| < \left| \sum_{i=1}^n f(x_i^*) (g(x_i) - g(x_{i-1})) \right| + \epsilon$$

$$\leq \sum_{i=1}^n \left| f(x_i^*) \right| \left| g(x_i) - g(x_{i-1}) \right| + \epsilon$$

$$\leq M \mathcal{V}(P, P) + \epsilon$$

$$\leq M \mathcal{V}_{a,b} f + \epsilon$$