Entropy, mixing, and independence

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Joint work with Hanfeng Li
Let \((X, \mu)\) be a probability space. Two sets \(A, B \subseteq X\) are independent if 
\[\mu(A \cap B) = \mu(A)\mu(B).\]

Suppose that we have a \(\mu\)-preserving action of a group \(G\) on \(X\). By considering the function
\[g \mapsto \mu(gA \cap B) - \mu(gA)\mu(B)\]
on \(G\) one can develop various notions of asymptotic independence as dynamical expressions of indeterminism or randomness:

- ergodicity
- weak mixing
- mixing
- completely positive entropy
In topological dynamics, the appropriate notion of independence is the combinatorial (or set-theoretic) one.

**Definition.** Let $X$ be a set. A collection $\{(A_{i,0}, A_{i,1})\}_{i=1}^n$ of pairs of subsets of $X$ is said to be independent if $\bigcap_{i=1}^n A_{i,\sigma(i)} \neq \emptyset$ for every $\sigma \in \{0, 1\}^{\{1,\ldots,n\}}$.

\[
\begin{array}{ccc}
  i = 1 & 0 & 1 \\
  i = 2 & 0 & 1 & 0 & 1 \\
  i = 3 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

In Banach space theory independence provides a means of constructing isomorphs of $\ell_1$ and as such plays an important role in Rosenthal’s $\ell_1$ theorem.
Let \((X, G)\) be a topological dynamical system. To what extent can independence be observed when we generate a family of pairs of subsets of \(X\) by applying the action of \(G\) to an initial pair \((A_0, A_1)\)?
In the context of entropy, we are concerned with independence over subsets of orbits with positive density. The topological entropy of a homeomorphism $T : X \to X$ measures the asymptotic exponential growth produced by applying $T$ iteratively to open covers:

$$h_{\text{top}}(T, U) = \lim_{n \to \infty} \frac{1}{n} \ln N(U \lor T^{-1}U \lor \cdots \lor T^{-n+1}U),$$

$$h_{\text{top}}(T) = \sup_U h_{\text{top}}(T, U)$$

where $N(\cdot)$ denotes the minimal cardinality of a subcover.
The local theory of entropy was initiated by Blanchard about 15 years ago via the notion of entropy pair. A pair \((x, y) \in X \times X\) is an \textit{entropy pair} if \(h_{\text{top}}(T, \mathcal{U}) > 0\) for every open cover \(\mathcal{U}\) consisting of the complements of disjoint open neighbourhoods of \(x\) and \(y\). Entropy tuples in \(X^k\) can be defined similarly.
Blanchard and Lacroix constructed the largest zero entropy factor of the system \((X, T)\), called the Pinsker factor, by taking the smallest closed invariant equivalence relation containing all entropy pairs. The system \((X, T)\) has

- *completely positive entropy* if every nontrivial factor has positive entropy (i.e., the Pinsker factor is trivial)
- *uniformly positive entropy* if every nondiagonal pair in \(X \times X\) is an entropy pair
- *uniformly positive entropy of all orders* if for each \(k \geq 2\), every nondiagonal tuple in \(X^k\) is an entropy tuple
Remarkably, every significant result to date involving entropy pairs (e.g., the product formula due to Glasner) has been obtained using measure-dynamical techniques by way of a variational principle.

**Problem.** Find more direct topological-combinatorial arguments.

This is one motivation for our approach to the local theory of entropy and mixing based on independence.

**Definition.** We call a tuple $x = (x_1, \ldots, x_k) \in X^k$ an *IE-tuple* if for every product neighbourhood $U_1 \times \cdots \times U_k$ of $x$ the orbit of the tuple $(U_1, \ldots, U_k)$ has an independent subset of positive density.
Using a local variational principle, Huang and Ye recently showed that entropy tuples are the same as nondiagonal IE-tuples. Inspired by work of Mendelson and Vershynin, we established a Sauer-Shelah-type coordinate density lemma that provides a combinatorial proof of this equivalence and applies in a universal way to other situations:

- entropy for actions of amenable groups
- sequence entropy for actions of arbitrary groups
- the property of tameness, which involves the presence of $\ell_1$ along infinite subsets of orbits of functions in the spirit of Rosenthal’s $\ell_1$ theorem

It is thus fruitful to define IN-tuples (independence along arbitrary large finite subsets of orbits) and IT-tuples (independence along infinite subsets of orbits) for the study of sequence entropy and tameness, respectively.
So the analysis of positive entropy production can be completely localized to the neighbourhood scale.

**Question.** How does the theory translate when consider locality in the dual sense at the $C^*$-algebra level of functions on $X$?

We have seen already that independence is connected with the presence of $\ell_1$. The link between topological entropy and $\ell_1$ structure via coordinate density was discovered by Glasner and Weiss, who used techniques from the local theory of Banach spaces to prove:

**Theorem** (Glasner-Weiss, 1995). If the homeomorphism $T : X \to X$ has zero entropy then so does the induced weak* homeomorphism of the space of probability measures on $X$. 
The missing ingredient for the systematic development of the connection to Banach space geometry observed by Glasner and Weiss is Voiculescu’s notion of approximation entropy:

**Definition.** Let $A$ be a unital nuclear $C^*$-algebra. Given a finite subset $\Omega \subset A$ and $\delta > 0$, we write $\text{rcp}(\Omega, \delta)$ for the infimum of $d$ over all diagrams

![Diagram](https://example.com/diagram.png)

which approximately commute to within $\delta$ on $\Omega$. For an automorphism $\alpha$ of $A$ we then define

$$ht(\alpha, \Omega) = \sup_{\delta > 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{rcp}(\Omega \cup \alpha \Omega \cup \cdots \cup \alpha^{n-1} \Omega, \delta),$$

$$ht(\alpha) = \sup_{\Omega} ht(\alpha, \Omega).$$
The Pinsker algebra is the invariant unital $C^*$-subalgebra of $C(X)$ corresponding to the Pinsker factor.

**Theorem** (K.-Li). For $f \in C(X)$ the following are equivalent:

1. $f$ is an element of the Pinsker algebra,
2. $\text{ht}(\alpha_T, \{f\}) > 0$,
3. there is an IE-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$,
4. there is a positive density set $I \subset \mathbb{Z}$ such that $\{f \circ T^k\}_{k \in I}$ is equivalent to the standard basis of $\ell_1$.

In particular we see that the system $(X, T)$ has completely positive entropy if and only if the orbit of every nonscalar $f \in C(X)$ has a positive density subset equivalent to the standard basis of $\ell_1$.

**Question.** What are the functional-analytic meanings of uniformly positive entropy and uniformly positive entropy of all orders?
What corresponds to combinatorial independence at the $C^*$-algebra level is tensor product independence.

**Theorem** (K.-Li). The system $(X, T)$ has uniformly positive entropy of all orders (i.e., every tuple is an IE-tuple) if and only if for every finite set $\Omega \subseteq C(X)$ and $\delta > 0$ there is a finite-dimensional unital subspace $V \subseteq C(X)$ with $\Omega \subseteq_\delta V$ such that the span of the products of the subspaces in a positive density subset $J$ of the orbit of $V$ is canonically isomorphic to $V \otimes J$.

For uniformly positive entropy we have a similar result, only now requiring the subspaces $V$ to be 2-dimensional.

We can thus recast our the theory of independence in terms of tensor products and extend its scope to general $C^*$-dynamical systems using the language of operator space theory.
**Problem.** Can an analogous theory of combinatorial independence be developed in measurable dynamics?

It frequently happens that combinatorial independence is present but not in a robust enough way to be measure-theoretically meaningful.

**Idea:** Observe whether combinatorial independence occurs to the appropriate degree in orbits of tuples of subsets whenever we hide from view a small portion of the ambient space at each stage of the dynamics.

For a topological system \((X, T)\) with \(T\)-invariant probability measure \(\mu\) we can thus define measure IE-tuples as in the topological case but subject to this control on our observations.
What is relevant now at the algebra level is Voiculescu’s von-Neumann-algebraic approximation entropy:

**Definition.** Let $M$ be a hyperfinite von Neumann algebra with faithful normal state $\sigma$. Given a finite subset $\Omega \subset M$ and $\delta > 0$, we write $\text{rcp}_\sigma(\Omega, \delta)$ for the infimum of $d$ over all diagrams

\[
\begin{array}{ccc}
M & \xrightarrow{\text{id}} & M \\
\phi & \downarrow \text{u.c.p.} & \downarrow \psi \\
M_d & \xrightarrow{\text{u.c.p.}} & M
\end{array}
\]

with $\sigma = \sigma \circ \psi \circ \phi$ which approximately commute in the $\sigma$-norm to within $\delta$ on $\Omega$. For an automorphism $\alpha$ of $M$ we then define

\[
\begin{align*}
\text{hcp}_\sigma(\alpha, \Omega) &= \sup_{\delta > 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{rcp}_\sigma(\Omega \cup \alpha \Omega \cup \cdots \cup \alpha^{n-1} \Omega, \delta), \\
\text{hcp}_\sigma(\alpha) &= \sup_{\Omega} \text{hcp}_\sigma(\alpha, \Omega).
\end{align*}
\]
Let $T$ be a measure-preserving automorphism of the probability space $(X, \mu)$. The Pinsker von Neumann algebra is the invariant unital von Neumann subalgebra of $L^\infty(X, \mu)$ corresponding to the largest zero entropy factor of $T$.

**Theorem (K.-Li).** Let $(X, T)$ be a topological system with ergodic $T$-invariant probability measure $\mu$. For $f \in L^\infty(X, \mu)$ the following are equivalent:

1. $f$ is an element of the Pinsker von Neumann algebra,
2. $\text{hcpa}_\mu(T, \{f\}) > 0$,
3. every $L^2$ perturbation of the orbit of $f$ contains a subset of positive density which is equivalent to the standard basis of $\ell_1$, and, in the case that $f \in C(X)$,
4. there is a measure IE-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$. 

**Corollary.** For a measure-preserving system \((X, \mu, T)\) the following are equivalent:

1. \((X, \mu, T)\) has complete positive entropy,
2. the orbit of every nonscalar \(f \in L^\infty(X, \mu)\) contains a subset of positive density which is equivalent to the standard basis of \(\ell_1\),
3. the induced homeomorphism of the spectrum of \(L^\infty(X, \mu)\) has uniformly positive entropy of all orders.

A *joining* between two systems \((Y, \nu, S)\) and \((Z, \omega, T)\) is an invariant probability measure on \(Y \times Z\) with \(\nu\) and \(\omega\) as marginals. The two systems are said to be *disjoint* if \(\nu \times \omega\) is the only joining between them. Viewing joinings as equivariant unital positive maps \(L^\infty(Y, \nu) \to L^\infty(Z, \omega)\), the above corollary gives a linear-geometric explanation for the disjointness of zero entropy systems and completely positive entropy systems.
In the context of sequence entropy we can define measure IN-tuples and a sequence version $\text{hcpa}_\mu^s(\cdot)$ of Voiculescu’s approximation entropy.

**Theorem** (K.-Li). Let $(X, G)$ be a topological system with ergodic $G$-invariant probability measure $\mu$. For $f \in L^\infty(X, \mu)$ the following are equivalent:

1. $f$ is an element of the maximal null von Neumann algebra,
2. $\text{hcpa}_\mu^s(X, \{f\}) > 0$ for some sequence $s$ in $G$,
3. every $L^2$ perturbation of the orbit of $f$ contains an infinite subset which is equivalent to the standard basis of $\ell_1$,
4. every $L^2$ perturbation of the orbit of $f$ contains arbitrarily large finite subsets which are $\lambda$-equivalent to the standard basis of $\ell_1$ for some $\lambda > 0$,

and, in the case that $f \in C(X)$,

5. there is a measure IN-pair $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$. 
**Corollary.** For a measure-preserving system \((X, \mu, T)\) the following are equivalent:

1. \((X, \mu, T)\) is weakly mixing.
2. The orbit of every nonscalar \(f \in L^\infty(X, \mu)\) contains an infinite subset which is equivalent to the standard basis of \(\ell_1\).
3. Every tuple for the induced action of \(G\) on the spectrum of \(L^\infty(X, \mu)\) is an IN-tuple,
4. Every tuple for the induced action of \(G\) on the spectrum of \(L^\infty(X, \mu)\) is an IT-tuple.