August,2009

Title, Author, Journal

An Extremal Sparsity Property of the Jordan Canonical Form

Richard A. Brualdi, Pei Pei, Xingzhi Zhan

Linear Algebra and Its Applications, 2008; 429 (10)

Outline

* Background Knowledge;
  - Jordan canonical form, similarity, sparsity, zero pattern.
* Main Results:
  - We prove that among all the matrices that are similar to a given square complex matrix, the Jordan canonical form has the largest number of off-diagonal zero entries.
  - We also characterize those matrices that attain this largest number.
* Further Discussion:
  - Real Jordan canonical form.

Similarity

Definition: A matrix $B \in M_n$ is said to be similar to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.

Remark:

* "Similarity" is an equivalence relation on M_n;
  - reflexive;
  - symmetric;
  - transitive.
* We can classify matrices by similarity.

Canonical form

* If we can find a set of "simple" matrices of prescribed form and see if both given matrices can be reduced by similarity to one of these simple forms. Such a set of representatives is often called a "Canonical form."
* There are three possible candidates for our "Canonical form."
  - Upper Triangular matrices;
  - Diagonal matrices;
  - Jordan matrices.

Upper Triangular matrices

* Every complex matrix A is similar to an upper triangular matrix whose diagonal entries (the eigenvalues of A) may be arranged in any given order.
* However, there is no uniqueness about the up triangular form.

Diagonal matrices

* If each of two given matrices is similar to a diagonal matrix, then they are indeed similar to each other if and only if the two diagonal matrices have the same main diagonal entries, counting multiplicities but ignoring order.
* However, not every complex matrix can be diagonalized.

For example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Jordan canonical form

* Every square **c**omplex matrix A is similar to a Jordan matrix

$$J(A) = diag(J_{n_1}(\lambda_1), \ldots, J_{n_k}(\lambda_k)),$$

where

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}$$

is a Jordan block of order n_i respect to eigenvale $\lambda_i, i = 1, \cdots, k$.

Example of Jordan canonical form

$$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J(A) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

* All the linear algebraic information about A is known at a glance.
* $J(A)$ is almost a diagonal matrix.
* $J(A)$ is unique up to permutations.

Symbols

* $J(A)$ is any of the Jordan canonical forms of A.
* $S(A)$ is the set of all complex matrices that are similar to A.
* $\sigma(A)$ is number of off-diagonal nonzero entries of A.

For example:

$$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J(A) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Conjecture

* Does $J(A)$ have the largest number of zero entries among all matrices in $S(A)$?
* The answer is no, as shown by the following example:

$$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J(A) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

* A has 11 zero entries while $J(A)$ has 10 zero entries.

Main result

Theorem1: Let A be a square complex matrix and $B \in S(A)$. Then

$$\sigma(B) \geq \sigma(J(A)).$$

proof: Method of Contradiction while using Lemma 1.

Jordan canonical form

* Every square **c**omplex matrix A is similar to a Jordan matrix

$$J(A) = diag(J_{n_1}(\lambda_1), \ldots, J_{n_k}(\lambda_k)),$$

where

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}$$

is a Jordan block of order n_i respect to eigenvalue $\lambda_i, i = 1, \cdots, k$.

Main result

Theorem1: Let A be a square complex matrix and $B \in S(A)$. Then

$$\sigma(B) \geq \sigma(J(A)).$$

proof: Assume A is of order n, $B \in S(A)$, and $J(A)$ has exactly k Jordan blocks. But

$$\sigma(B) < \sigma(J(A)) = n - k.$$

Then by lemma 1, $J(B)$ has at least $k+1$ Jordan blocks which is contrary to $B \in S(A)$.

Main result

Lemma1: Let n, k be positive integers with $1 \leq k \leq n$. If a matrix A of order n satisfies $\sigma(A) \leq n - k$, then there exists a permutation matrix P such that

$$P^T AP = diag(A_1, A_2, \ldots, A_k)$$

where A_j is square and non-void for $j = 1, \cdots, k$.

Main result

Proof of Lemma1: Let $A = [a_{ij}]$. With A we associate a graph G with vertex set $V = 1, 2, \cdots, n$ where there is an edge between vertices i and j if and only if $i \neq j$, and $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Main result

Proof of Lemma1: This graph G has p edges where $p \leq \sigma(A) \leq n-k$. We list the edge of G in some order $e_1, e_2, \ldots, e_p$.
Let G_i be the graph with vertex set V and edges $\{e_1, e_2, \ldots, e_i\}, i = 0, 1, \cdots, p$.
Thus G_0 has no edges, $G_p = G$, and G_i is obtained from G_{i-1} by including the new edge $e_i, i = 1, 2, \cdots, p$.

Main result

Proof of Lemma1: It follows that G_i has at most one fewer connected component than G_{i-1}, since one edge can at best join together two connected components. Since G_0 has n (trivial) connected components, G_p has at least $n - p \geq n - (n - k) = k$ connected components $C_1, C_2, \ldots, C_{n-p}$. Let $C_k^{'}$ be the union of the connected components C_j with $j \geq k$. The principal submatrices $A_1, \ldots, A_{k-1}, A_k$ of A corresponding to $C_1, \ldots, C_{k-1}, C_k^{'}$ satisfy the conclusion of the lemma.□

Main result

Theorem2: Let A be a square complex matrix and $B \in S(A)$. Then $\sigma(B) = \sigma(J(A))$ if and only if there exists a monomial matrix M such that

$$M^{-1}BM = J(A).$$

Remark: A square complex matrix is called a monomial matrix if it has exactly one nonzero entry in each row and each column.
proof: Method of induction on order n while using Lemma 2.

Main result

Lemma2: Let

$$A = \begin{bmatrix} a & x^T \\ 0 & B \end{bmatrix}$$

be a complex matrix of order n where B has order $n-1$. If $J(A)$ has only one Jordan block, then $J(B)$ has only one Jordan block.

* Theorem 2 tells us that $J(A)$ is the unique zero-non zero pattern among $S(A)$ that attains the largest number of off-diagonal zero entries.
* **Monomial Matrix** If Γ_n be the set of Monomial Matrices of order n, then $M \in \Gamma_n \iff \exists$ a permutation matrix P and a nonsingular diagonal matrix D such that

$$M = PD \quad \text{or} \quad M = DP.$$

* Γ_n is a multiplicative group.

Main result

* **Conclusion:** Theorem 1 and Theorem 2 show that up to permutation similarity, $J(A)$ is the unique zero-nonzero pattern among matrices in $S(A)$ that attains the largest number of off-diagonal zero entries.
* **Question:** Whether a conclusion similar to Theorem 1 and Theorem 2 holds for the real Jordan canonical form?
* **The answer is no.**

Further Discussion

* Theorem 1 and Theorem 2 are not true for real Jordan canonical form (see example 1).
* **But** Theorem 1 is true for real, diagonalizable matrices and their real Jordan canonical forms.
* **However,** Theorem 2 is not true for real, diagonalizable matrices and their real Jordan canonical forms (see example 2).

Further Discussion

Example 1: Theorem 1 and Theorem 2 are not true for real Jordan canonical forms.

$$RJ(A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

where $TAT^{-1} = RJ(A)$

$$T = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Further Discussion

Example 2: Theorem 2 is not true for real, diagonalizable matrices and their real Jordan canonical forms.

$$RJ(A) = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Further Discussion

Jordan canonical form:
Each square real matrix $A \in M_n(R)$ is similar to a block diagonal real matrix of the form:

$$RJ(A) = \begin{bmatrix} C_{n_1}(a_1, b_1) & & & & & 0 \\ & \ddots & & & & \\ & & C_{n_p}(a_p, b_p) & & & \\ & & & J_{n_q}(\lambda_q) & & \\ & & & & \ddots & \\ 0 & & & & & J_{n_r}(\lambda_r)) \end{bmatrix}$$

where $\lambda_k = a_k + ib_k, k = 1, 2, \cdots, p.$

Further Discussion

Jordan canonical form:

$$C_k(a,b) = \begin{bmatrix} C(a,b) & I & & 0 \\ & C(a,b) & \ddots & \\ & & \ddots & I \\ 0 & & & C(a,b) \end{bmatrix}$$

where $C(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$