MATH 817 Notes

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<u>composition series</u>

<u>normal series</u>

<u>refinement</u> of a normal series

<u>equivalence</u> of two normal series (of same group)

<u>Lemma</u> Any two normal series of a group G admit equivalent refinements.

<u>Theorem</u>[J-H] Any two composition series of a group G are equivalent.

Why does Lemma $\Rightarrow$ Theorem?

Pf of Theorem granting Lemma:

Say $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_\ell = G$ and $\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_k = G$ are two composition series.

By the lemma, they have equivalent refinements. The quotients for the refinement of the 1st are, in some order,

$$N_0/N_1, N_2/N_1, \ldots, N_\ell/N_{\ell-1}, \underbrace{\{e\}, \{e\}, \ldots, \{e\}}_{\text{some \# of copies}}$$

and the quotients of the refinement of the 2nd are

$$M_0/M_1, M_2/M_1, \ldots, M_\ell/M_{\ell-1}, \{e\}, \ldots, \{e\}.$$

Since N_i/N_{i-1} and M_j/M_{j-1} are not trivial, it follows $\ell = k$ and

$$N_i/N_{i-1} = M_{\sigma(i)}/M_{\sigma(i)-1}, \exists \sigma \in S_\ell.$$

$\square$

<u>Sketch of Pf of Lemma</u>: Say $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_\ell = G$ and $\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_k = G$ aare any two <u>normal</u> series. Refine the 1st by replacing $N_i \trianglelefteq N_{i+1}$ with

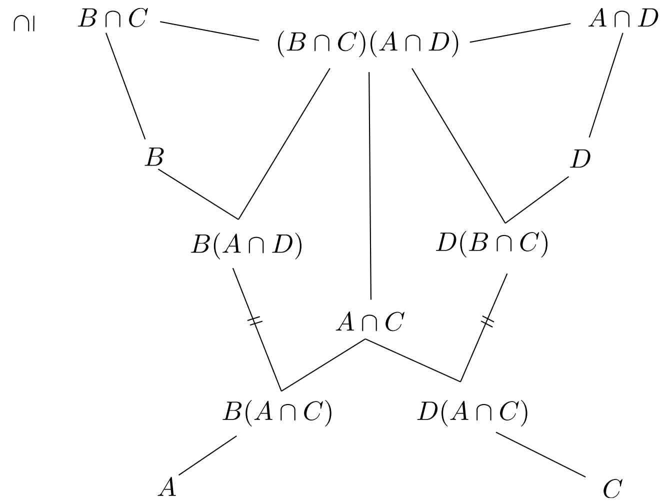
$$\underbrace{N_i \cdot (N_{i+1} \cap M_0)}_{= N_i} \trianglelefteq N_i \cdot (N_{i+1} \cap M_1) \trianglelefteq \ldots \trianglelefteq \underbrace{N_i \cdot (N_{i+1} \cap M_k)}_{= N_{i+1}}, \forall i = 0, 1, \ldots, \ell - 1$$

It's true that each inclusion really is $\trianglelefteq$ and each of these really is a subgroup of G.

Do the same with roles reversed: Replace $M_j \trianglelefteq M_{j+1}$ with

$$\underbrace{M_j \cdot (M_{j+1} \cap N_0)}_{= M_j} \trianglelefteq M_j \cdot (M_{j+1} \cap N_1) \trianglelefteq \ldots \trianglelefteq \underbrace{M_j \cdot (M_{j+1} \cap N_k)}_{= M_{j+1}}$$

<u>Zassenhaus Butterfly Lemma</u>: $B \trianglelefteq A$ $D \trianglelefteq C$, $A, C \leq G$



$B(A \cap D) \trianglelefteq B)A \cap C)$, $D(B \cap C) \trianglelefteq D(A \cap C)$, and

$$\boxed{\frac{B(A \cap C)}{B(A \cap D)} \cong \frac{D(A \cap C)}{D(B \cap C)}}$$

The quotients of 1st refinement are:

$$\frac{N_i(N_{i+1} \cap M_{j+1})}{N_i(N_{i+1} \cap M_j)} \leftarrow \quad \begin{array}{l} i = 0, \dots, \ell - 1 \\ j = 0, \dots, k - 1 \end{array}$$

and of the 2nd are:

$$\frac{M_j(M_{j+1} \cap N_{i+1})}{M_j(M_{j+1} \cap N_i)} \leftarrow \quad \begin{array}{l} i = 0, \dots, \ell - 1 \\ j = 0, \dots, k - 1 \end{array}$$

For a given i, j, these are isomporhic by Zassenhaus' Lemma.

<u>Theorem</u> [Feit-Thompson] Every group of odd order is solvable.

Fix n. $\mathbb{R}[x_1, x_2, \dots, x_n]$ = ring of all $\mathbb{R}$-valued polynomial function in n variables.

e.g. $3x^2 - \pi yz \in \mathbb{R}[x, y, z]$

S_n acts on $\mathbb{R}$ via

$$\sigma \cdot f(x_1, \dots, x_m) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \ \forall \sigma \in S_n, \forall f \in \mathbb{R}(x_1, \dots, x_n)$$

$\sigma = (1\ 2\ 3)$ $\sigma \cdot (3x_1^2 - \pi x_2 x_3) = 3x_2^2 - \pi x_3 x_1$

- $\tau \cdot (\sigma \cdot f) = (\tau\sigma) \cdot f$
- $e \cdot f = f$

So, it's a group action.

In fact, S_n acts on $\mathbb{R}(x_1, \dots, x_n)$ as an <u>algebra</u>:

$*$ $\sigma \bullet (f \cdot g) = (\sigma \bullet f) \cdot (\sigma \bullet g)$

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- $\sigma \bullet (f+g) = \sigma \bullet f + \sigma \bullet g$
- $\sigma \bullet (rf) = r(\sigma \bullet f)$

$\forall \sigma \in S_n, f, g \in \mathbb{R}[x_1, \ldots, x_n], r \in \mathbb{R}$

Fix n. Define

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in \mathbb{R}[x_1, \ldots, x_n]$$

e.g. $n = 3$: $\begin{aligned} \Delta &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ &= x_1^2 x_2 \ldots \end{aligned}$

Consider $\sigma \cdot \Delta = \prod_{i<j} \sigma \cdot (x_i - x_j) = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})$

<u>Lemma</u> $\sigma \cdot \Delta = \pm \Delta, \forall \sigma \in S_n$