MATH 817 Notes

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I'll post the final version of problem set #4 today (due Wednesday)

Recall:

- G simple if $G \neq \{e\}$ and $\{e\}$ and G are the only normal subgroups of G.
- A composition series of G is a chain
$$e = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_\ell = G$$
such that $\ell \geq 0$ and ${N_i}/{N_{i-1}}$ is simple $\forall i = 1, \ldots, \ell$

Note N_i need not be normal in G

Prop. If $\#G < \infty$ then it has a composition series.

Pf:

- If $G = \{e\}$, $\ell = 0$ works: $\{e\} = N_0 = G$.
- $\#G > 1$. $S = \{N \mid N \trianglelefteq G, N \neq G\}$
S is non-empty since $\{e\} \in S$. So, let $N \in S$ be maximal (S is a nonempty, finite poset).

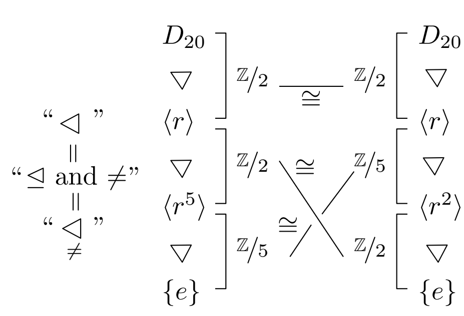
Then $\#N < \#G$ and by induction on order, N has a composition series:
$$\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_j = N.$$

I claim $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq \overset{N}{\overset{\parallel}{N_j}} \trianglelefteq N_{j+1} = G$ is a composition series of G. It suffices to show ${G}/{N}$ is simple.

${G}/{N} \neq \{e\}, N \neq 0$.

If $\mathcal{L} \underset{\neq}{\trianglelefteq} {G}/{N}$ is a proper normal subgroup, then $\mathcal{L} = {H}/{N}$ for some $H \leq G$, $N \leq H$ and $H \underset{\neq}{\trianglelefteq} G$, by the Lattice Theorem (4th Isomorphism Theorem). So $H \in \mathcal{L}$ and $N \leq H \Rightarrow H = N \Rightarrow \mathcal{L} = \left\{e_{G/N}\right\}$. $\therefore$ ${G}/{N}$ is simple. ∎

Ex (1)



$\ell = 3 = k \ \sigma = (1\ 2)$

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Theorem [Jordan-Holder] If $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_\ell = G$ and $\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_k = G$ are two composition series of the same group G, then $\ell = k$ and $\exists \sigma \in S_\ell$ such that

$$N_i/N_{i-1} \cong M_{\sigma(i)}/M_{\sigma(i)-1} \ \forall i = 1, 2, \ldots, \ell.$$

Ex (2)

$$\begin{array}{ll}
\mathbb{Z}/2 & \left[\begin{array}{l} S_4 \\ \triangledown\vert \\ A_4 \end{array}\right. \\
\mathbb{Z}/3 & \left[\begin{array}{l} \triangledown\vert \\ V = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \end{array}\right. \\
\mathbb{Z}/2 & \left[\begin{array}{l} \triangledown\vert \\ \langle (1\ 2)(3\ 4) \rangle \end{array}\right. \\
\mathbb{Z}/2 & \left[\begin{array}{l} \triangledown\vert \\ \{e\} \end{array}\right.
\end{array}$$

Def: If G has a composition series,

$$\{e\} = N_0 \trianglelefteq \ldots \trianglelefteq N_\ell = G,$$

the multi-set $\{N_1/N_0, N_2/N_1, \ldots, N_\ell/N_{\ell-1}\}$ is the multi-set of composition functions of G. It's well defined by Jordan-Holder.

A group is solvable if it has a composition series and its composition factors are all abelian groups (and thus are all $\mathbb{Z}/p$, for various primes p.)

Note: If G us simple and abelian, then $G \cong \mathbb{Z}/p$ for some prime p.

Pf: Pick $x \in G$, $x \neq e$, then $e \neq \langle x \rangle \trianglelefteq G$ and so $G = \langle x \rangle$. It follows $\#G$ must be prime.

Ex D_{20}, S_4 are solvable.

If G is solvable, then $\#G = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ and the composition factos of G are

$$\overbrace{\mathbb{Z}/p_1, \cdots, \mathbb{Z}/p_1}^{e_1}, \overbrace{\mathbb{Z}/p_2, \cdots, \mathbb{Z}/p_2}^{e_2}, \ldots, \overbrace{\mathbb{Z}/p_n, \cdots, \mathbb{Z}/p_n}^{e_n}$$

Why? If $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_\ell = G$ and $\#G$, then

$$\#G = \# N_1/N_0 \cdot \# N_2/N_1 \cdots \# N_\ell/N_{\ell-1}$$

Ex $S_n, n \geq 5$

composition functions of S_n: $A_n, \mathbb{Z}/2$

Since A_n is simple, $A_n \trianglelefteq S_n$,

$$S_n/A_n \cong \mathbb{Z}/2 \quad \{e\} \trianglelefteq A_n \trianglelefteq S_n$$

Def A normal series of a group G is any chain $\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots \trianglelefteq N_\ell = G$. ($N_i$ need not be normal in G, N_i/N_{i-1} need not be simple, $N_i = N_{i-1}$ is allowed)

A refinement of a given normal series of G is any pther normal series of G obtained by "inserting more steps"

Two normal series $e = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_k = G$ and $e = M_0 \trianglelefteq M_1 v \ldots \trianglelefteq M_\ell = G$ are equivalent if $k = \ell$ and $\exists \sigma \in S_\ell$ such that

$$M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}, i = 1, \ldots, \ell$$

Prop Any two normal series for the same group admit equivalent refinements.