# MATH 817 Notes

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Problem Set 2:
Prelim version posted soon
Due Thursday (9-10)

Def If $G$ is a group + $X$ is a set, an action of $G$ on $X$ is a function $G \times X \to X$. usually written

$$(g, x) \mapsto g \cdot x \in X$$

s.t.

$$g' \cdot (g \cdot x) = (g' \cdot g) \cdot x \quad \forall g, g' \in G, x \in X$$

(1) action; group action

(2) $e \cdot x = x, \forall x \in X$

Ex (1) $F$ = any field e.g. $F = \mathbb{R}$

$GL_n(F)$ acts on $F^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_i \in F \right\}$ via

$$\underset{(n\times n)}{A} \cdot \underset{(n\times 1)}{\vec{v}} = \underset{\uparrow n\times 1 \text{ matrix mult}}{A\vec{v}} \in F^n$$

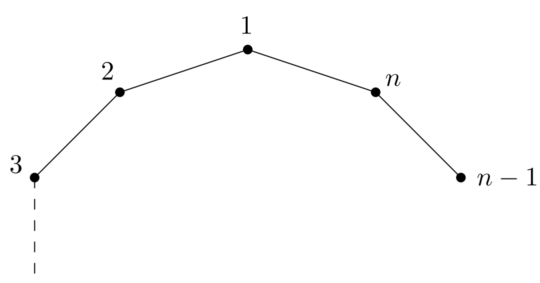
$\forall A \in GL_n(F)\ \forall \vec{v} \in F^n$

- $A' \cdot (A \cdot \vec{v}) = (A'A) \cdot \vec{v}$ ✓
- $I_n \vec{v} = \vec{v}$

(2) $S_n$ acts on $X = \{1, s, \dots, n\}$ via $\sigma \cdot x = \sigma(x)$

- $\sigma' \cdot (\sigma \cdot x) = (\sigma' \circ \sigma)(x)$ ✓
- $e \cdot x = x$ ✓

(3) $D_{2n}$ acts $X = \{1, 2, \dots, n\}$ as follows: Label vertices of $P_n$ $1, 2, \dots, n$ + for $g \in D_{2n}$, $g \cdot i = j$ where $j$ is the vertex that $g$ sends $i$ to.



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$$r \cdot i = \begin{cases} i+1 & 1 \le i \le n-1 \\ 1 & i = n \end{cases}$$

$$\begin{aligned} s \cdot 1 &= 1 \\ s \cdot 2 &= n \\ s \cdot 3 &= n-1 \\ &\vdots \end{aligned}$$

<u>Prop</u> Let $G$ be a group + $X$ a set.

- Given an action $\cdot$ of $G$ on $X$, the function $\varphi : G \to \text{Perm}(X)$ defined by $\varphi(g)(x) = g \cdot x$ is a group hom.
- Given a group hom. $\varphi : G \to \text{Perm}(X)$, the pairing $g \cdot x := \varphi(g)(x)$ is a group action
- These constructions are mutual inverses:

$$\{\text{actions of } G \text{ on } X\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} \{\text{group hom's } G \to \text{Perm}(X)\}$$

<u>Pf of 1st claim</u> Given $g', g \in G$,

$$\begin{aligned} \forall x, \varphi(g'\, g)(x) &= (g'g) \cdot x \\ &\overset{①}{=} g' \cdot (g \cdot x) \\ &= g' \cdot \varphi(g)(x) \\ &= \varphi(g')((\varphi(g)(x)) \\ &= (\varphi(g)' \circ \varphi(g))(x) \end{aligned}$$

$\therefore\ \varphi(g'\, g) = \varphi(g') \circ \varphi(g)$.

$\therefore\ \varphi$ is a group hom.

<u>WAIT</u>:

We need to check: $\forall g \in G$

$$(x \mapsto g \cdot x) \in \text{Perm}(X)$$

<u>Proof</u>:
$\varphi(g) \cdot \varphi(g^{-1}) = \varphi(g \cdot g^{-1}) = \varphi(e)$

$+\ \varphi(e)(x) = e \cdot x \overset{②}{=} x$

$\therefore \varphi(g) \circ \varphi(g^{-1}) =$ identity map on $X$.

Similarly, we can show $\varphi(g^{-1}) \circ \varphi(g) =$ ID map.

$\therefore\ \varphi(g)$ is 1-1 and onto and $\varphi(g)^{-1} = \varphi(g^{-1})$

<u>More Ex's</u>

④ $G =$ any group, $X := G$

$G$ acts on $X$ via "left multiplication"

$$\underset{\substack{\uparrow \\ \text{action}}}{g \cdot x} := \underset{\substack{\uparrow \\ \text{group law}}}{gx}$$

$g' \cdot (g \cdot x) = (g'g) \cdot x$
$e \cdot x = x$

If $\#G = n < \infty$, by Prop, the left action of $G$ on itself yield a group hom.

$$\varphi : G \to \text{Perm}(G) \overset{\substack{\text{choose a numbering of } G \\ \downarrow}}{\cong} S_n$$

e.g. $G = S_3 = \left\{ \overset{1}{e}, (1\ \overset{2}{1}\ 2), (1\ \overset{3}{\ }\ 3), (2\ \overset{4}{\ }\ 3), (1\ \overset{5}{2}\ 3), (1\ \overset{6}{3}\ 2) \right\}$

$\varphi : S_3 \to S_6$

$\bullet \varphi(e) = e$

$\bullet \varphi((1\ 2)) = (1\ 2)(3\ 6)(4\ 5)$

$(1\ 2) \cdot \overset{1}{e} = (1\ \overset{2}{\ }\ 2)$

$(1\ 2) \cdot (1\ \overset{2}{\ }\ 2) = \overset{1}{e}$

$(1\ 2) \cdot (1\ \overset{3}{\ }\ 3) = (1\ \overset{6}{3}\ 2)$

$(1\ 2) \cdot (2\ \overset{4}{\ }\ 3) = (3\ \overset{5}{1}\ 2)$

$(1\ 2) \cdot (1\ \overset{5}{2}\ 3) = (3\ \overset{4}{\ }\ 2)$

$(1\ 2) \cdot (1\ \overset{6}{3}\ 2) = (1\ \overset{3}{\ }\ 3)$

can repeat for all elements of $G$

Note $\varphi : G \to \text{Perm}(G) \cong S_n$, $n = \#G$ is 1-1.

$\therefore$ Every finite group is isomorphic to a subgroup of $S_n$, some $n$.

(5) $G$ = any group, $X := G$

$G$ acts on $X$ "by conjugation":

Define $g \cdot x = gxg^{-1}$

$$\begin{aligned} g' \cdot (g \cdot x) &= g'(gxg^{-1})(g')^{-1} \\ &= (g'g)x(g'g)^{-1} \\ &= (g'g) \cdot x \checkmark \end{aligned}$$

$(ab)^{-1} = b^{-1}a^{-1}$

$e \cdot x = exe^{-1} = x$

If $G$ is abelian, this action (by conjugation) is the trivial action $g \cdot x = x\ \forall\ \forall g$

The trivial action of $G$ on $X$ corresponds to the trivial homomorphism $\varphi : G \to \text{Perm}(X)$.

$$\varphi(g) = e_{\text{Perm}(X)}\ \forall g.$$

Def A Subgroup of a group $G$ is a subset $H$ of $G$ s.t.

(1) $e \in H$

(2) If $x, y \in H$ then $xy \in H$. (3) If $x \in H$, then $x^{-1} \in H$.

Notation $H$ is a subgroup of $G \Leftrightarrow H \leq G$

Lemma Given a subset $H$ of a group $G$,

$$H \leq H \quad \Leftrightarrow \quad \begin{array}{l} \text{(a) } H \neq \varnothing \\ \text{(b) } \forall x, y \in H, xy^{-1} \in H. \end{array}$$

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<u>Pf</u> ($\Rightarrow$) clear

($\Leftarrow$) Check 3 axioms:

(1) (a) $\Rightarrow \exists x \in H$ + so by (b)

$$x \cdot x^{-1} \in H \;\therefore e \in H$$

(3) $e \in H \overset{(b)}{\Rightarrow}$ if $y \in H, y^{-1} = ey^{-1} \in H$.
(2) If $x, y \in H$, then $y^{-1} \in H$ by (3). $\therefore$ by (b)

$$x \cdot (y^{-1})^{-1} \in H \;\therefore xy \in H$$

<u>Examples</u>

(1) $\{e\} \leq G$

(2) $G \leq G$

(3) $\left\{1, r, r^2, \ldots, r^{n-1}\right\} \leq D_{2n}$

$\{1, 2\} \leq D_{2n}$

(4) Pick $m \leq n$

Let $H = \left\{ \sigma \in S_n \mid \begin{array}{c} \sigma(m+1) = m+1 \\ \sigma(m+2) = m+2 \\ \vdots \sigma(n) = n \end{array} \right\}$

$H \leq S_n$

$H \cong S_m$