MATH 817 Notes

JD Nir
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/817.html
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- I'll return pset 2 to your boxes
- pset 3 due today

<u>Lagrange</u>: $\#G < \infty$.

If $H \leq G$, then $\#H \mid \#G$.

Does converse hold? No

Variation: Does

If $d \mid \#G$, then $\exists H \leq G$ s.t. $\#H = d$.

hold?

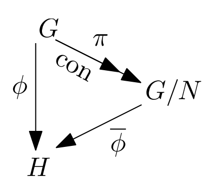
No. But:

- If G is cyclic of finite order, then if $d \mid \#G$, then $\exists! H \leq G$ with $\#H = d$.
- If G is finite, $p = d$ is prime, and $p \mid \#G$, then $\exists x \in G$ s.t. $|x| = p$, + thus $\#\langle x \rangle = p$ [Cauchy]
- G finite, p prime, if $p^m \mid \#G$ but $p^{m+1} \nmid \#G$, then $\exists H \leq G$ with $\#H = p^m$ [Sylow]

(we will prove these)

<u>Theorem</u> [Universal mapping property of a quotient group]

If $N \trianglelefteq G, \varphi : G \to H$ is a group homomorphism such that $N \leq \ker \varphi$, then $\exists!$ group homomorphism $\overline{\varphi} : {}^{G}/_{N} \to H$ such that

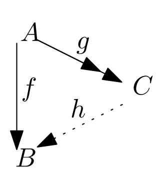


commutes; i.e.

$\overline{\varphi} \circ \pi = \varphi$ where $\pi : G \to {}^{G}/_{N}$ is the canonical map.

<u>Pf</u> Define $\overline{\varphi} : {}^{G}/_{N} \to H$ by $\overline{\varphi}(g \cdot N) = \varphi(g)$.

- Well-defined: $gN = g'N \Rightarrow g'g^{-1} \in N \subseteq \ker \varphi$
  $\Rightarrow \varphi(g'g^{-1}) = e \Rightarrow \varphi(g') \cdot \varphi(g)^{-1} = e \Rightarrow \varphi(g') = \varphi(g)$
- $\overline{\varphi}(xN \cdot yN) = \overline{\varphi}(xyN) = \varphi(xy) = \varphi(x) \cdot \varphi(y) = \overline{\varphi}(xN) \cdot \overline{\varphi}(yN) \therefore$ $\overline{\varphi}$ is a group homomorphism.
- $\overline{\varphi} \circ \pi = \varphi$ is obvious from definition of $\overline{\varphi}$
- $\overline{\varphi}$ is unique since π is onto. □



A, B, C sets
g onto

JD Nir
jnir@huskers.unl.edu

$\forall c \in C, \exists a$ such that $c = g(a)$

If $h \circ g = f$, then $h(g(a)) = f(a) \Rightarrow h(a) = f(a)$.

Lemma $\varphi : G \to H$ is a group homomorphism.

φ is 1-1 $\Leftrightarrow \ker \varphi = \{e_G\}$

Pf: $(\Rightarrow)$ $\varphi(e_G) = e_H$ ✓

$(\Leftarrow)$ Say $\varphi(x) = \varphi(y)$. Then $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} = e \Rightarrow xy^{-1} = e \Rightarrow x = y$.

Theorem [1st Isomorphism Theorem for Groups]

Given a group homomorphism $\varphi : G \to H$, the map

$$\overline{\varphi} : G/\ker\varphi \to H$$

given by

$$\overline{\varphi}(g \cdot \ker\varphi) = \varphi(g)$$

is an injective group homomorphism with $\operatorname{im}\overline{\varphi} = \operatorname{im}\varphi$. So $\boxed{G/\ker\varphi \cong \operatorname{im}\varphi}$

Pf: UMP with $N = \ker\varphi$ gives $\exists$ group homomorphism $\overline{\varphi} : G/\ker\varphi \to H$, given by $\overline{\varphi}(g \cdot \ker\varphi) = \varphi(g)$.

$\operatorname{im}\overline{\varphi} = \operatorname{im}\varphi$ is obvious

$\overline{\varphi}(x \cdot \ker\varphi) = e_H \Rightarrow \varphi(x) = e_H \Rightarrow x \in \ker\varphi$

$\Rightarrow x \cdot \ker\varphi = \ker\varphi = e_{g/\ker\varphi}$. $\therefore$ By Lemma, $\overline{\varphi}$ is 1-1. $\square$

① $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ is an onto group homomorphism with kernel $SL_n(\mathbb{R})$.

$$\uparrow \quad \left(\forall x, \det\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = x\right)$$

$\therefore GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^\times$ via $A : SL_n(\mathbb{R}) \mapsto \det(A)$

$$\begin{bmatrix} r & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \cdot SL_n(\mathbb{R}) \mapsfrom r$$

② Define $\varphi : (\mathbb{R}, +) \to \mathbb{C}^\times$ by $\varphi(t) = e^{2\pi i t}$

φ is a group homomorphism by laws of exponents.

$\operatorname{im}\varphi = \{z \mid ||z|| = 1\}$ = unit circle in $\mathbb{C}$

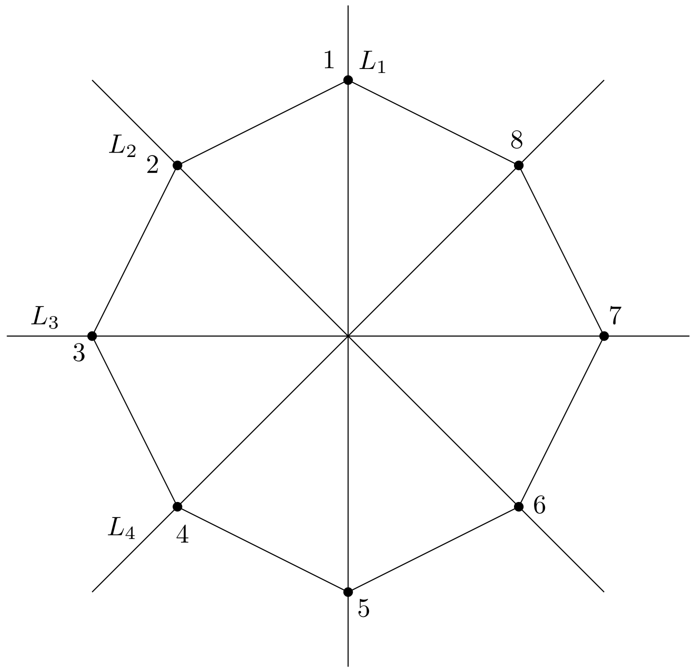
$\ker\varphi = \mathbb{Z} \leq \mathbb{R}$

$$\therefore \mathbb{R}/\mathbb{Z} \cong \{z \mid ||z|| = 1\} \leq \mathbb{C}^\times$$

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & & \{z \mid ||z||=1\} \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \mathbb{Q}/\mathbb{Z} & \cong & \{z \mid z^n = 1, \text{some } n\} \end{array}$$

JD Nir
jnir@huskers.unl.edu

(3)



$L_1 L_2 L_3 L_4$ = lines of symmetry passing through verticies

G acts on $\{L_1, L_2, L_3, L_4\}$ and hence $\exists$ group homomorphism $\varphi : G \to S_4$.

E.g.

$\varphi(r) = (1\ 2\ 3\ 4)$

$\varphi(s) = (2\ 4)$

$\operatorname{im} \varphi = \langle (1\ 2\ 3\ 4), (2\ 4) \rangle \leq S_4$

$\ker \varphi = \{e, r^4\} = Z(D_{16})$

$\therefore\ \frac{D_{16}}{Z(D_{16})} \cong \langle (1\ 2\ 3\ 4), (2\ 4) \rangle \leq S_4$

In fact, $\frac{D_{16}}{Z(D_{16})} \cong D_8$

$D_{16} = \langle r, s \mid r^8, s^2, srsr \rangle$

${}^{D_{16}}/_{\{e, r^4\}} = \langle r, s \mid \cancel{r^8}, s^2, srsr, r^4 \rangle = D_8$

(4) Define $\varphi : \mathbb{C}^\times \to \mathbb{C}^\times$ by $\varphi(z) = z^{12}$.

- $\varphi(zw) = (zw)^{12} = z^{12} w^{12} = \varphi(z) \cdot \varphi(w)$
- $\operatorname{im} \varphi = \mathbb{C}^\times$, by Fundamental Theorem of Algebra
- $\ker \varphi = \left\{1, e^{\frac{2\pi i}{12}}, e^{2\pi i \frac{2}{12}}, \ldots, e^{2\pi i \frac{11}{12}}\right\}$

$$\therefore \frac{\mathbb{C}^\times}{\{z \mid z^{12} = 1\}} \cong \mathbb{C}^\times$$

$w \cdot \ker \varphi \mapsto w^{12}$