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MATH 817 Notes

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- Final version of problem set 3 is posted
- I'll have pset 2 graded by tomorrow

Recall If $N \trianglelefteq G$, then the equivalence relation $x \sim y \Leftrightarrow xN = yN \Leftrightarrow Nx = Ny$ respects multiplication and thus $G/N := G/\sim$

Exercise: If $\sim$ represents multiplication then $\sim$ "comes from" some $N \trianglelefteq G$.

Ex Say $G = \langle x \rangle$ is cyclic.

What can we say about quotients of G?

$N \trianglelefteq G \Leftrightarrow N \leq G$ (G is abelian) and $N \leq G \Rightarrow N = \langle x^m \rangle$, some m

$\therefore\ G/N = \{N, xN, x^2N, \ldots, x^{m-1}N\}$ (if $m \neq 0$) and $G/N = \langle xN \rangle$ is cyclic

$\therefore$ Quotient groups of cyclics are cyclic

Fact $\#\frac{\langle x \rangle}{\langle x^m \rangle} = \begin{cases} m & |x| = \infty \\ \frac{|x|}{\gcd(m,|x|)} & |x| < \infty \end{cases}$

Ex $\mathbb{Q} = (\mathbb{Q}, +)$

$\mathbb{Z} \trianglelefteq \mathbb{Q}$

$\mathbb{Q}/\mathbb{Z} = \{q + \mathbb{Z} \mid 0 \leq q < 1, q \in \mathbb{Q}\}$

$q\mathbb{Z} \neq q'\mathbb{Z}$ for $0 \leq q < q' < 1$

$q = \frac{m}{n}\ m, n \in \mathbb{Z}, n > 0$

$n(q + \mathbb{Z}) = m + \mathbb{Z} = \mathbb{Z}$

$\therefore [q + \mathbb{Z}] \leq n$ in $\mathbb{Q}/\mathbb{Z}$

$\mathbb{Q}/\mathbb{Z} \cong \{z \mid z \in \mathbb{C}^\times, z^n = 1 \text{ for some } n\} \leq \mathbb{C}^\times$ via $q + \mathbb{Z} \mapsto e^{2\pi i q}$ (well defined since $e^{2\pi i n} = 1\ \forall n \in \mathbb{Z}$)

$\mathbb{R} = (\mathbb{R}, +)$

$\mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C}^\times \mid ||x|| = 1\} \leq \mathbb{C}^\times$

$r + \mathbb{Z} \mapsto e^{2\pi i r}$

Lagrange

Lemma $H \leq G$. $\forall x, y \in G\ \#(xH) = \#(yH)$ and $\#(Hx) = \#(Hy)$

(Valid even if those are infinite)

Pf Define $\alpha : xH \to yH$ by

$$\alpha(xh) = (yx^{-1})(xh) = yh, \forall h \in H$$

Define $\beta : yH \to xH$ by $\beta(yh) = xh$

Then $(\alpha \circ \beta)(yh) = yh\ \forall h$

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$(\beta \circ \alpha)(xh) = xh\ \forall h.\ \therefore\ \#xH = \#yH$

Langrange's Theorem If $H \leq G$ and $\#G < \infty$ then $\#H \mid \#G$ and $\frac{\#G}{\#H}$ = the number of left cosets of H in G and the number of cosets of H in G.

Pf The left cosets of H in G partition G into subsets each of size $\#H$.

$G = g_1H \cup g_2H \cup \ldots \cup g_\ell H$, for some $g_1, \ldots, d_\ell \in G$ with $g_iH \cap g_jH = \varnothing\ \forall i \neq j$

By lemma, $\#g_iH = \#g_jH = \#H$

$\therefore\ \#G = \ell \cdot \#H$ and ℓ = the number of left cosets of H in G.

Similarly, $G = Hy_1 \cup \ldots \cup Hy_r$, $r = \#$ of right cosets and $\#Hy_i = \#H\ \forall i$

$\therefore\ \#G = r \cdot \#H$ □

Note: In any group H, given $H \leq G$

$\#$ of left cosets of H = $\#$ of right cosers of H

$\{xH \mid x \in G\} \xleftrightarrow[\text{onto}]{\text{1-1}} \{Hy \mid y \in G\}$

$$\begin{array}{ccc} xH & \mapsto & Hx^{-1} \\ y^{-1}H & \leftarrow\!\shortmid & Hy \end{array}$$

$xH = x'H \Rightarrow (x')^{-1}x \in H \Rightarrow H(x')^{-1} = Hx^{-1}$

Def $H \leq G$, $[G : H] :=$ # of left cosets of H in G ($\overset{\text{note}}{=}$ # of right cosets)

It's called the index of H in G.

Lagrange: $H \leq G$, $\#G < \infty \Rightarrow \#G = \#H \cdot [G : H]$

$([G : H] = \frac{\#G}{\#H})$

Cor If $x \in G$ and $\#G < \infty$, then $|x| \mid \#G$

Pf $\langle x \rangle \leq G$ and $\#\langle x \rangle = |x|$. Apply Lagrange. □

Cor If G is a group of prime order then G is cyclic. In fact, $G = \langle x \rangle$ for any $x \in G, x \neq e$.

Prop If $H \leq G$ and $[G : H] = 2$ then $H \trianglelefteq G$. (Valid even if $\#G = \infty$)

Pf: Pick any $g \in G \setminus H$. Then $H \cap gH = \varnothing$. So, $[G : H] = 2$ $G = H \cup gH$.

Likewise $G = H \cup Hg$ (by the Note).

$\therefore\ gH = G \setminus H = Hg$ and so $gH = Hg\ \forall g \in G \setminus H$

If $g \in H, gH = H = Hg$. $\therefore\ gH = Hg\ \forall g \in G$. $H \trianglelefteq G$. □

Ex $\rangle r \rangle \leq D_{2n}$

$$\begin{array}{ccc} \#\langle r \rangle & \overset{\text{Lagrange}}{\Longrightarrow} & [D_{2n} : \langle r \rangle] = 2 \\ & \Rightarrow & \langle r \rangle \trianglelefteq D_{2n} \end{array}$$

Note If $\#G < \infty$ and p is the mallest prime divisor of $\#G$, then if $H \leq G$ and $[G : H] = p$, then $H \trianglelefteq G$.