MATH 817 Notes

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Defintion

Let G be a group, $S \subseteq G$ is a subset of G.

$$\begin{aligned} \langle S \rangle &= \text{“subgroup of } G \text{ generated by } S\text{”} \\ &= \bigcap_{\substack{H \le G \\ S \subseteq H}} H \end{aligned}$$

If $K \le G$ and $S \subseteq K$, then $\langle S \rangle \le K$.

Example

If $H, K \le G$, then $\langle H \cup K \rangle$ = the unique smallest subgroup of G that contains H and K.

“Explicit” description of $\langle S \rangle$:

$$\langle S \rangle = \left\{ s_1^{e_1} s_2^{e_1} \cdots s_\ell^{e_\ell} \mid \ell \ge 0, s_i \in S, e_i \in \mathbb{Z} \right\}$$

Example

$r, s \in D_{2n}.\langle r, s \rangle = D_{2n}$

In general we can have $r^{e_1} s^{e_2} r^{e_3} = r^{e_1 - e_3} s$ or $r^{e_1 + e_3}$

Example

$G = GL_2(\mathbb{R})$. Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y = \left(\begin{smallmatrix} 0 & 2 \\ 1/2 & 0 \end{smallmatrix}\right)$.

$x^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $y^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so $|x| = 2, |y| = 2$.

Then $\langle x, y \rangle \le G$ and $\#\langle x, y \rangle = \infty$

$$xy = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \in \langle x, y \rangle$$

and

$$(xy)^n = \begin{bmatrix} 1/2^n & 0 \\ 0 & 2^n \end{bmatrix} \ne I_2 \ \forall n \ne 0$$

Example

Let $S = \{\text{transpositions in } S_n\} \subseteq S_n$

$\langle S \rangle = S_n$

Let $S = \{(a\ b\ c) \in S_n\}$. What is $\langle S \rangle =$? A_n?

Let $H \le G$ and $K \le G$. Then

$$HK := \{hk \mid h \in H, k \in K\} \underset{}{\overset{\swarrow \text{subset}}{\subseteq}} G$$

Lemma

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If $H \subseteq N_G(K)$ or if $K \subseteq N_G(H)$ then $HK = KH$ and $HK \leq G$.

(So in this case, $HK = \langle H \cup K \rangle$)

Proof

Assume $K \subseteq N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. So $kHk^{-1} = H$ $\forall k \in K$. If $hk \in HK$, then

$$\begin{aligned} hk &= kk^{-1}hk \\ &= k\underbrace{(k^{-1})h(k^{-1})^{-1}} \\ &= kk^{-1}H(k^{-1})^{-1} \\ &= kH \in KH \end{aligned}$$

If $kh \in KH$, then $kh = \underbrace{khk^{-1}}_{H} k \in HK$

$\therefore$ $KH = HK$

Check $HK \leq G$:

- $e = ee \in HK$ ✓
- $h_1k_1, h_2k_2 \in HK \Rightarrow h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = \underbrace{h_1k_1k_2^{-1}(k_2k_1^{-1})}_{\in H}(k_1k_2^{-1})h_2^{-1} \in HK$

$\therefore$ $HK \leq G$

The rest holds by symmetry. □

Example

Let $H = \langle r \rangle \leq D_{2n}$, $K = \langle s \rangle \leq D_{2n}$

$K \subseteq N_G(H)$ since $sr^is^{-1} = sr^is = r^{-i} \in H$

$\therefore$ $HK \leq D_{2n}$

In this example, $HK = D_{2n}$

Quotient Groups

Let G be a group. Suppose $\sim$ is an equivalence relation on G:

(i) $x \sim x$

(ii) $x \sim y \Rightarrow y \sim x$

(iii) $x \sim y$ and $y \sim z \Rightarrow x \sim z$

$\sim$ respects multiplication provided: if $x \sim y$, then $x \cdot z \sim y \cdot z$ $\forall z \in G$ and $w \cdot x \sim w \cdot y$ $\forall w \in G$.

For $x \in G$, given $\sim$, define $[x] = \{y \in G \mid y \sim x\}$ = equivalence class of X.

For $x, y \in G, [x] = [y] \Leftrightarrow x \sim y$

x is not $\sim y \Leftrightarrow [x] \cap [y] = \varnothing$

Lemma

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If $\sim$ respects multiplication, then the set "${}^{G}/{\sim}$" $= \{[x] \mid x \in G\}$ of all equivalence classes is a group under $[x] \cdot [y] := [xy]$. Moreover, the cannonical function $G \to {}^{G}/{\sim}$ (namely, $g \to [x]$) is a surjective group homomorphism.

Proof

• The rule is well-defined (that is, independent of choices for x, y). If $[x] = [x']$ and $[y] = [y']$, then I claim $[xy] = [x'y']$:

$$\begin{array}{lccc} & [x] = [x'] & \Rightarrow & x \sim x' \\ & [y] = [y'] & \Rightarrow & y \sim y' \\ \Rightarrow & xy \sim x'y & \text{and} & x'y \sim x'y' \\ \Rightarrow & xy \sim x'y' & & \\ \Rightarrow & [xy] \sim [x'y'] & \checkmark & \end{array}$$

• if $([x][y])[z] = [xy][z] = [(xy)z] = [x(yz)] = [x][yz] = [x]([y][z])$

• $[e][x] = [ex] = [x]$
$[x][e] = [xe] = [x] \;\therefore\; [e] = e_{G/\sim}$

• Given $[x] \in {}^{G}/{\sim}$ $\begin{array}{l} [x][x^{-1}] = [xx^{-1}] = [e] = e_{G/\sim} \\ [x^{-1}][x] = [x^{-1}x] = [e] = e_{G/\sim} \end{array}$

$\therefore$ ${}^{G}/{\sim}$ is a group. The map $G \to {}^{G}/{\sim}$, $x \to [x]$ is a group homomorphism.

Example

$G = (\mathbb{Z}, +)$. Fix n, define $\sim$ on G by $i \sim j \Leftrightarrow i = j \bmod n$

$i \sim j \Rightarrow i + \ell \sim j + \ell$

$\therefore$ ${}^{Z}/{\sim}$ is a group. ${}^{\mathbb{Z}}/{\sim} := {}^{\mathbb{Z}}/n = \{[0], [1], \ldots, [n-1]\}$

Definition

Let G be a group. $H \leq G$ is a subgroup.

A left coset of H in G is a subset of the form xH for some $x \in G$.

$xH = \{xh \mid h \in H\}$

A right coset of H in G is a subset of the form Hx for some $x \in G$.

Lemma

$H \leq G . \forall x, y \in G$

• either $xH = yH$ or $xH \cap yH = \varnothing$
• either $Hx = Hy$ or $Hx \cap Hy = \varnothing$