MATH 817 Notes

JD Nir
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/817.html
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Problem Set #1 is due tomorrow, by 5pm.

Group homomorphisms.

Ex

$$\varphi : D_{2n} \to (\{\pm 1\}, \cdot)$$

$\varphi(g) = \begin{cases} 1 & \text{if } f \text{ is orientation preserving} \\ -1 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{reversing} \end{cases}$

$\varphi(g \circ h) = \varphi(g) \cdot \varphi(h)$

$\varphi(r^i) = 1$

$\varphi(sr^i) = -1$

Ex sgn : $s_n \to (\{\pm 1\}, \cdot)$

$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ can be writter as a product of an even \# of transpositions} \\ -1 & \text{else} \end{cases}$

$\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma) \cdot \text{sgn}(\tau)$

If $\sigma = (1\ 2)(3\ 4), \text{sgn}(\tau) = -1$. So, $\tau = (a_1\ b_1) \dots (a_{2i+1}\ b_{2i+1})$

Then $\sigma \circ \tau = \underbrace{(1\ 2)(3\ 4)(a_1\ b_1) \dots (a_{2i+1}\ b_{2i+1})}_{2i+3 \text{ is odd}}$

Fact If τ is a product of an odd # of transpositions, then $\underbrace{\text{it cannot be expresssed as a product of an even \#}}_{\text{sgn}(\tau)=-1}$

$\sigma \in S_n$ Associated to σ we have a permutation matrix:

$$P_\sigma := \begin{array}{c} \\ 1 \\ \vdots \\ \sigma(2) \\ \\ \sigma(1) \\ \\ n \end{array} \overset{\begin{array}{cccc} 1 & 2 & \cdots & n \end{array}}{\begin{pmatrix} 0 & 0 & & \\ \vdots & 0 & & \\ & 1 & & \\ 0 & 0 & & \\ 1 & \vdots & & \\ 0 & & & \\ 0 & 0 & & \end{pmatrix}_{n \times n}} \in GL_n(F)$$

Fact: $\det(P_\sigma) = \text{sgn}(\sigma)$

$$\begin{array}{ccc} S_n & \overset{\varphi}{\to} & GL_n(\mathbb{R}) \\ \downarrow \text{sgn} & & \downarrow \det \\ (\{\pm 1\}, \cdot) & \subseteq & (\mathbb{R}^x, \cdot) \end{array} \qquad \begin{array}{l} \varphi(\sigma) := P_\sigma \\ \varphi \text{ is a group homomorphism} \\ + \text{ this square commutes.} \end{array}$$

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Def G, H groups. A group isomorphism from G to H is a homomorphism $\varphi : G \to H$ that is 1-1 + onto.

Lemma If $\varphi : G \to H$ is a group isomorphism, so is $\varphi^{-1} : H \to G$

Pf NTS $\varphi^{-1}(a \cdot b) \overset{(*)}{==} \varphi^{-1}(a) \cdot \varphi^{-1}(b), \forall a, b \in H$.

Note $\begin{aligned} \varphi(\varphi^{-1}(a \cdot b)) = a \cdot b &= \varphi(\varphi^{-1}(a)) \cdot \varphi(\varphi^{-1}(a)) \\ &= \varphi(\varphi^{-1}(a) \cdot \varphi^{-1}(b)) \end{aligned}$, since φ is a homomorphism.

Since φ is 1-1, $(*)$ holds

Def Two groups $G + H$ are isomorphic if $\exists$ an isomorphism from$G$ to H (or vice versa).

Ex (1) $\exp(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$ is an isomporhism.

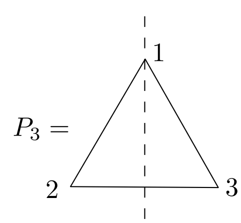
$\exp(x) = e^x$

It's inverse is $\ln : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$

$\exp \circ \ln =$ identity on $\mathbb{R}_{>0}$

$\ln \circ \exp =$ " " $\mathbb{R}$

(2) $D_6 \simeq S_3$. Define $\varphi : D_6 \to S_3$ by $\varphi(y) =$ the permutation of $\{1, 2, 3\}$ given by what g does to the verticies of P_3.

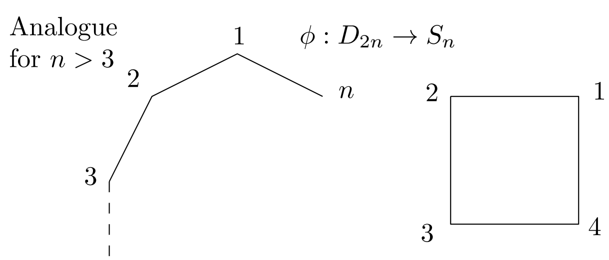


e.g. $\varphi(r) = (1\ 2\ 3)$

$\varphi(s) = (2\ 3)$

If $\varphi(g) = \varphi(h)$ g and h do the same thing to the vertices and hence $g = h$.

$\therefore$ φ is 1-1 + thus onto because $\#D_6 = \#S_3$



Is $D_{24} \simeq S_4$? φ won't work

(3) $GL_2(\mathbb{F}_2) \simeq S_3$, not obvious

$\mathbb{F}_2 =$ field with 2 elements, $\mathbb{F}_2 = \mathbb{Z}/2$

(4) Are D_8 and Q_8 isomorphic?

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

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No! Say $\varphi : Q_8 \to D_8$ were an isomorphism.

Then $\varphi(i), \varphi(-1), \varphi(j), \varphi(-j), \varphi(k), \varphi(-k)$ would be 6 elements of D_8 each having order 4.

$\Rightarrow\Leftarrow$

Lemma If $\varphi : G \to H$ is an isomorphism of groups,

$$|\varphi(g)| = |g| \ \forall g \in G$$

Pf: Omitted

If $G + H$ are isomorphic, then they share all "group theoretic" properties.

- cardinality of groups
- carfinality of centers
- # of elements of a given order

(5) If x is any set,

$$(S_X =) \operatorname{Perm}(X) = \{\sigma : X \to X \mid \sigma \text{ is a 1-1 + onto function}\}$$

Then Perm is a group under $\circ$.

E.g. $X = \{1, 2, \ldots, n\}, \operatorname{Perm}|X| = S_n$.

If $f : X \to Y$ is a 1-1 and onto funciton between sets X and Y, then $\varphi : \operatorname{Perm}(X) \to \operatorname{Perm}(Y)$ defined by

$$\varphi(\sigma) = f \circ \sigma \circ f^{-1}$$

is a group isomporhism.