MATH 817 Notes

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2nd version of 1st problem set is posted. Final revision today. To hand it in, print off (or write) solutions + put it in my box/hand it to me.

G group, $x \in G$, $n \in \mathbb{Z}$

$$x^n = \begin{cases} \overbrace{x \cdots x}^{n} & n > 0 \\ e & n = 0 \\ (x^{-1})^{|n|} & n < 0 \end{cases}$$

Lemma 1 $x^i \cdot x^j = x^{i+j} \forall i, j \in \mathbb{Z}$

2

$S_n =$ bijections from $\{1, 2, \ldots, n\}$ to itself

It's a group under $\circ$

If $\sigma \in S_n$, support$(\sigma) := \{i \in \{1, \ldots, n\} \,|\sigma(i) \neq i\}$

e.g. $(1\ 3\ 7) \in S_8$

support$((1\ 3\ 7)) = \{1, 3, 7\}$

If $\sigma, \tau \in S_n$, $\sigma + \tau$ are disjoint if

$$\text{support}(\sigma) \cap \text{support}(\tau) \neq \varnothing$$

$\sigma \circ \tau = \tau \circ \sigma$

Prop 1 The order of an m-cycle is m.

2 Disjoint permutations commute.

Theorem Every permutation is a product of pairwise disjoint cycles + such a factorization is unique up to ordering.

Ex:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 2 & 5 & 1 & 6 \end{pmatrix} \in S_7$$

$$\begin{aligned} \sigma &= (1\ 3\ 7\ 6) \circ (2\ 4) \\ &= (2\ 4) \circ (1\ 3\ 7\ 6) \end{aligned}$$

Pf of existence Algorithm: Given $\sigma \in S_n$

- Pick a $\in$ min(support(σ))
- Let $m = \min \text{pos in s.t.} \sigma^m(a) = a$

- Form $(a\ \sigma(a)\ \sigma^2(a) \quad \sigma^{m-1}(a)) =: \tau_1$
- Let $b = \min(\text{support}(\sigma) \setminus \{a, \sigma(a), \dots, \sigma^{n-1}(a)\})$
- form $(b\ \sigma(b)\ \cdots \sigma^{n-1}(b)) =: \tau_2$
  $n = $ min pos int s.t. $\sigma^n(b) = b$
- Continue until φ
  Then $\sigma = \tau_1 \circ \tau_2 \circ \dots$

Notes 1 $e =$ empty product

2 $(1\ 2\ 3) = (3\ 1\ 2) = (2\ 3\ 1)$

these are the same elt of S_n, using different notation

Prop If $\sigma = \tau_1 \circ \tau_2 \cdots \tau_k$, τ_i's are pairwise disjoint cycles, then

$$|\sigma| = \operatorname{lcm}\{|\tau_1|, \dots, |\tau_k|\}$$

Pf: By HW, $\sigma^n = \tau_1^n \circ \cdots \circ \tau_k^n, \forall n$

Claim: $\tau_1^n \circ \cdots \circ \tau_k^n = e \Leftrightarrow \tau_i^n = e, \forall i$

($\Leftarrow$) obvious

($\Rightarrow$) Note that $\tau_1^n, \dots, \tau_k^n$ are pairwise disjoint permutations. ETS if $\alpha_1, \dots, \alpha_k$ are pairwise disjoin + $\sigma_1, \dots, \sigma_k = e$, then $alpha_i = e\ \forall i$

Pick $a \in \text{support}(\alpha_i)$. Then $a = (\alpha_1 \cdots \alpha_k)(a) = \alpha_1(a)$.

$\therefore\ \alpha_1 = e$. So $\alpha_2 \cdots \alpha_n = e$ + thus $\alpha_2 = e, \dots \alpha_k = e$.

This proves the Claim.

So,

$$\begin{aligned}|\sigma| &= \min\{n \in \mathbb{N} \mid \tau_i^n = e, \forall i\} \\ &= \min\{n \in \mathbb{N} \mid |\tau_i| \mid n, \forall i\} =: \operatorname{lcm}\{|\tau_1|, \dots, |\tau_k|\}\end{aligned}$$

□

So far: $GL_n(F), D_{2n}, S_n$

One more basic example: $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$

- $e = 1$
- $ij = k, jk = i, ki = j$
- $(-1)(-1) = 1$
- $(-1)i = -i = i(-1)$, etc
- $i^2 = -1, j^2 = -1, k^2 = -1$

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$ij = k \Rightarrow (ij) = ik \Rightarrow (-1)j = ik \Rightarrow -j = ik$

Similarly, $\begin{array}{l} ik = -j \\ ji = -k \\ kj = -i \end{array}$

Realize Q_8 as a "subgroup" of $GL_2(\mathbb{C})$ = a group, $\mathbb{C} = \mathbb{R} + \mathbb{R} \cdot \sqrt{-1}$

Let $i := \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$

$j := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$k := \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$

$1 := I_2$ $-1 := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$-i = \begin{bmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{bmatrix}$ etc for $-j, -k$

Check that all the relations hold

$\boxed{Q_8\text{: Every subgroup is normal, but } Q_8 \text{ is not abelian}}$

Def'n If GH are two groups, a group homomorphism from$G$ to H is a function

$$\varphi : G \to H$$

s.t. $\varphi(\underset{\substack{\uparrow \\ \cdot = \text{mult. in } G}}{x \cdot y}) = \varphi(x) \underset{\substack{\uparrow \\ \cdot = \text{mult. in } H}}{\cdot} \varphi(y), \forall x, y \in G$

Lemma If $\varphi : G \to H$ is a group hom. then $\varphi(e_G) = e_H$.

Pf: $\varphi(e_G) = \varphi(e_G \cdot e_G) = \varphi(e_G) \cdot \varphi(e_G)$

If $x \in H$ + $x = x^2$, then $x^{-1} \cdot x = x^{-1}x^2$ + so $e_H = x$.

$\therefore$ $\varphi(e_G) = e_H$). $\square$

Ex 1 det: $GL_n(F) \to F^x$, F = field.

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

2 $\exp : (\mathbb{R}, +) \to \mathbb{R}^x$ $\exp(r) := e^r$

$$\exp(r + s) = \exp(r) \cdot \exp(s)$$

3 $\ln : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$

$$\ln(r \cdot s) = \ln(r) + \ln(s)$$