# MATH 817 Notes

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August 24, 2015

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This is Math 817.

Def: A monoid is a set $M$ equipped with a pairing

$$M \times M \to M$$

written $(a, b) \mapsto a \cdot b = ab$

Satisfying 2 axioms:
(1) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \forall a, b, c \in M$
(2) $\exists e \in M$ s.t. $e \cdot a = a = a \cdot e \forall a \in M$

Lemma If $(M, \cdot)$ is a monoid, then the element $e$ in (2) is unique:

$$e \cdot a = a = a \cdot e \ \forall a \in M + e' \cdot a = a = a \cdot e' \ \forall a \in M \Rightarrow e = e'$$

Pf: Let $e, e'$ be as above.

Then $e \cdot e' = e' + e = e \cdot e' \therefore e = e'$ □

Def: A group is a pair $(G, \cdot)$ s.t. $(G, \cdot)$ is a monoid + also
(3) $\forall x \in G, \exists y \in G$ s.t. $x \cdot y = e = y \cdot x$

Lemma: In axiom (3), $y$ is unique.

Pf: Given $x$, suppose $xy = e = yx + xz = e = zx$.
Then

$$\begin{aligned} &(z \cdot x) \cdot y = e \cdot y = y \\ + \ &z \cdot (x \cdot y) = z \cdot e = z \\ &\therefore y = z \text{ using } (1) \end{aligned}$$

Notation $x \in G$, $G$ group, $x^{-1}$ = the unique elt of axiom (3).

Additive notation
$+$ instead of $\cdot$
$-x$ instead of $x^{-1}$
0 instead of $e$

Def A group $G, \cdot)$ is abelian if
(4) $x \cdot y = y \cdot x \ \forall x, y \in G$

Ex
(1) $(\mathbb{Z}, +)$: is an abelian group
(1) $+$ is assoc.
(2) $e = 0$
(3) $n + (-n) = 0 \ \forall n$
(4): $n + m = m + n$

(2) $(\mathbb{R} \setminus \{0\}, \cdot)$ $e = 1$
is an abelian group

(3) $F$ = any field (e.g. $F = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{C}$ or $\mathbb{Q}/p$ or ...)
$n \in \mathbb{N} = \{1, 2, 3, 4 \dots\}$

$G(n|F) = \{n \times n \text{ matricies } A | \det A \neq 0\}$

Let $\cdot$ = matrix mult.

$\cdot$ is a pairing on $G(n|F)$
(1) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
(2) $e = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$
(3) if $\det(A) \neq 0$, $A^{-1}$ exists.

In fact, $A^{-1} = \frac{1}{\det(A)} A^{\text{adj}}$

$(A^{\text{adj}} A = A A^{\text{adj}} = \det(A) I_n)$

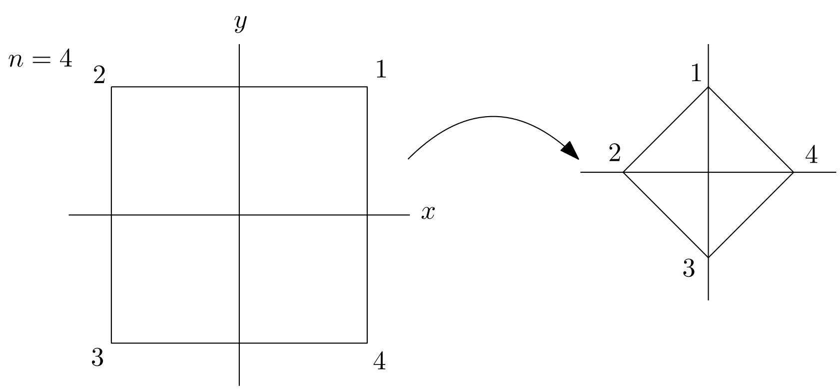
$\therefore (G(n|F), \cdot)$ is a group, It's not abelian unless $n = 1$.

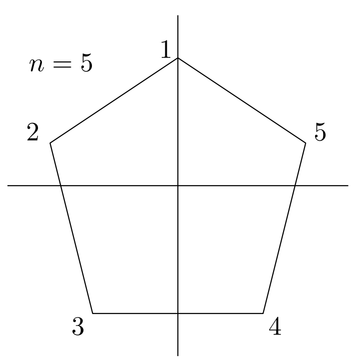
$(G(1|\mathbb{R}) = \mathbb{R} \setminus \{0\})$

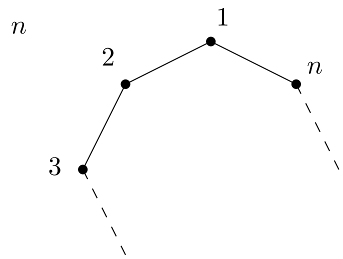
(4) $(\frac{\mathbb{Z}}{n}, +)$ $\frac{\mathbb{Z}}{n} = \left\{\overline{0}, \overline{1}, \dots, \overline{n-1}\right\}$ $\underbrace{\text{is an abelian group}}$

(5) $((\frac{\mathbb{Z}}{n})^x, \cdot)$ $(\frac{\mathbb{Z}}{n})^x = \left\{\overline{j} \in \frac{\mathbb{Z}}{n} | \gcd(j, n) = 1\right\}$ ditto

(6) $n \in \mathbb{N}, n \geq 3$ $D_{2n}$ = group of symmetries of a regular $n$-sided polygon in $\mathbb{R}^2$

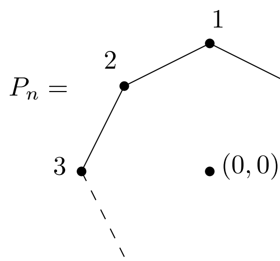





$\cdot$ = composition

$P_n$ = regular $n$-gon, centered at $(0,0)$, symmetric about $y$-axis




$D_{2n} = \{g : \mathbb{R}^2 \to \mathbb{R}^2 | g \text{ is an isometry } (g \text{ preserves distance + angles}) \text{ s.t. } g(P_n) = P_n\}$

$(D_{2n}, \cdot)$ is a group

$\cdot$ is a pairing on $D_{2n}$ ✓

(1) comp is assoc.

(2) the identity map on $\mathbb{R}^2$ is $e$

(3) isometries are $1-1$ and onto

So if $g \in D_{2n}, g^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ exists as a function. $g^{-1} \in D_{2n}$

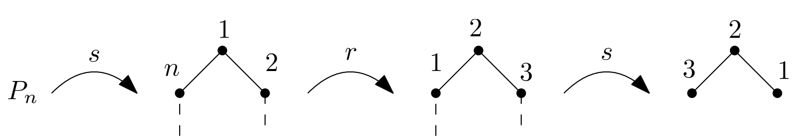
Two obvious elts on $D_{2n}$:

$r$ = rotation of $\mathbb{R}^2$ (centered at $(0,0)$) by $\frac{2\pi}{n}$ radians, counter-clockwise

$s$ = reflection about $y$-ais

$r, s \in D_{2n}$

- $r^n = e \Rightarrow r^{n-1} = r^{-1}$
- $s^2 = e \Rightarrow s^{-1} = s$
- $srs = ?$




$\boxed{srs = r^{-1}}$ and $\boxed{srsr = e}$
and
$\boxed{sr = r^{-1}s}$