MATH 817 Notes

JD Nir
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/817.html
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- Problem set #7 id due today
- We'll start linear algebra Monday
  - Dummit-Foote, Ch. 11
  - Friedberg-Insel-Spence, Linear Algebra, 4$^{\text{th}}$ ed

$K = \langle k \rangle$

$\rho_1, \rho_2 : K \to \operatorname{Aut}(H)$

$\sigma\rho_1\sigma^{-1} = \rho_2(K)$

$\sigma\rho_1(k)\sigma^{-1} = \rho_2(k)^a$

$$\underset{(h,k)}{H \rtimes_{\rho_1} K} \underset{\mapsto}{\to} \underset{(\sigma(h),k^a)}{H \rtimes_{\rho_2} K}$$

$\tau\rho_2(k)\tau^{-1} = \rho_1(k)^b$

$\#K < \infty \Rightarrow \ker\rho_1 = \ker\rho_2 = N$

$\Rightarrow \rho_1, \rho_2$ both factor as $K \to K/N \overset{\overline{\rho_i}}{\rightarrowtail} \operatorname{Aut}(H)$

$\#\rho_1(K) = \#\rho_2(K)$

$\Rightarrow \#\ker\rho_1 = \#\ker\rho_2$

$\Rightarrow \ker\rho_1 = \ker\rho_2$ (K is cyclic)

Prop Given $H, K, \rho : K \to \operatorname{Aut}(H)$, let $G = H \rtimes_\rho K$

1 $\tilde{H} := \{(h, e) \mid h \in H\} \trianglelefteq G$, $H \cong \tilde{H}$ via $h \mapsto (h, e)$.

2 $\tilde{K} := \{(e, k) \mid k \in K\} \leq G$, $K \cong \tilde{K}$ via $k \mapsto (e, k)$.

3 $\tilde{H} \cap \tilde{K} = \{e_G\}$

4 $\tilde{H} \cdot \tilde{K} = G$

5 The action of $\tilde{K}$ on $\tilde{H}$ via conjugation corresponds to ρ via $H \cong \tilde{H}$, $K \cong \tilde{K}$.

Pf 1 - 4 ommitted

5 $(e, k) \cdot (h, e) \cdot (e, k)^{-1} = (h^k, k) \cdot (e, k^{-1}) = (h^k \cdot e, k \cdot k^{-1}) = (h^k, e)$

$\rho(k)(h) = h^k$ □

Cor $H \rtimes_\rho K$ is abelian $\Leftrightarrow$ H is abelian, K is abelian and ρ is trivial.

Note ρ is trivial $\Leftrightarrow \tilde{K} \trianglelefteq G$

$\Leftrightarrow H \rtimes_\rho K = H \times K$.

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<u>Cor</u> Every automorphism of a group H occurs as conjugation by some element $g \in G$ where G is a group such that $H \trianglelefteq G$.

<u>Pf</u> Given $H, \alpha \in \mathrm{Aut}(H)$, let $G = H \rtimes_\rho \mathbb{Z}$, where $\rho : \mathbb{Z} \to \mathrm{Aut}(H)$

$\rho(1) = \alpha$

Then $H \trianglelefteq G$ and conjugation on H by $(e, 1) \in G$ is α. $\square$

classifying all groups of order $18 = 2 \cdot 3^2$

Say $\#G = 18$. $n_3 = 1$

$\exists H \trianglelefteq G, \#H = 9$

$\exists K \leq G, \#K = 2$ (Cauchy)

$K = \langle z \rangle, |z| = 2$

$G = H \rtimes_\rho K$, $\rho : K \to \mathrm{Aut}(H)$

$\alpha = \rho(z) \in \mathrm{Aut}(H)$

$\#H = 9 \Rightarrow H$ is abelian and $H \cong \mathbb{Z}/9$ or $H \cong \mathbb{Z}/3 \times \mathbb{Z}/3$

I: $H \cong \mathbb{Z}/9$ $H = \langle x \rangle, |x| = 9$

$\underset{\alpha}{\mathrm{Aut}(H)} \underset{\leftrightarrow}{\cong} \underset{1 \text{ or } -1}{(\mathbb{Z}/9)^\times}$

$\alpha = \mathrm{id}$ or $\alpha(x) = x^{-1}$

$G = \mathbb{Z}/9 \times \mathbb{Z}/2$ $\quad G = \langle x, z \mid x^9, z^2, zxz^{-1} = x^{-1} \rangle$

$\cong \boxed{D_{18}}$

II: $H \underset{\substack{x \leftrightarrow (1,0) \\ y \leftrightarrow (0,1)}}{\cong} \mathbb{Z}/\times\mathbb{Z}/3$ $H = \langle x, y \mid x^3, y^3, xyx^{-1}y^{-1} \rangle$

$$\begin{array}{ccc} \mathrm{Aut}(H) & \cong & GL_2(\mathbb{Z}/3) \\ \Downarrow & & \\ \alpha & \longleftrightarrow & A \end{array}$$

$\#\,\mathrm{Aut}(H) = 8 \cdot 6 = 2^4 \cdot 3$

$|\alpha| = 1 \Rightarrow \boxed{G = \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/2}$

$A^2 = I_2, A \neq I_2$ e.g. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ldots$

<u>Fact from linear algebra</u>

If $A \in GL_2(\mathbb{Z}/3)$ and $A^2 = I_2$,

then $A \sim \begin{bmatrix} 1 & - \\ 0 & 1 \end{bmatrix}$ or $A \sim \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or $A \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\#G = 75 = 3^2 \cdot 5$ $n_5 = 1$

$h \cong \mathbb{Z}/5 \times \mathbb{Z}/5 \Rightarrow \#\,\mathrm{Aut}(H) = 24 \cdot 20 = 2^5 \cdot 3 \cdot 5$

$K \cong \mathbb{Z}/3$

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$K = \langle z \rangle, |z| = 3$

$\rho : K \to \mathrm{Aut}(H)$

$$\rho(k) = \langle \rho(z) \rangle, \#\rho(K) = \begin{matrix} 1 \\ \text{or} \\ 3 \end{matrix}$$

If ρ, ρ' are any two non-trivial homomorphism from K to $\mathrm{Aut}(H)$, $\rho(K)$ and $\rho'(K)$ are conjguate since they are Sylow 3-subgroups.