MATH 817 Notes

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Def V, W finite dimensional

$\varphi \in \mathrm{Hom}_F(V, W)$

rank$(\varphi) = \dim(\mathrm{im}\, \varphi)$

nulity$(\varphi) = \dim(\ker \varphi)$

rank$(\varphi)+$ nulity$(\varphi) = \dim(V)$

e.g. $V = F^n, W = F^m, \varphi = T_A, A = m \times n$

rank$(A)+$ nulity$(A) = n = \#$ of columns

rank$(A) =$ dimension of column space

A is equivalent to $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$, $r = \mathrm{rank}\, A$.

Recall: V finite dimensional, $\varphi : \mathrm{Hom}_F(V, V)$

φ is an isomorphism iff $\varphi(B)$ is a basis for every basis B.

Pick $B = (v_1, v_2, \ldots, v_n)$, a basis.

$$\begin{aligned} \varphi \text{ is an isomorphism} &\Leftrightarrow (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_n) \text{ form a basis.} \\ &\Leftrightarrow \varphi(v_1), \varphi(v_2), \ldots, \varphi(v_n) \text{ are linearly independent} \\ &\Leftrightarrow M^B_B(\varphi) \text{ has linearly independent columns} \end{aligned}$$

Say $\#F < \infty$ (e.g., $F = \mathbb{Z}/p$, p prime)

Let $A \in \mathrm{Mat}_{n\times m}(F)$. A is invertible (i.e., $A \in GL_n(F)$)

$\Leftrightarrow$ the columns $c_1, c_2, \ldots, c_n$ of A are linearly independent

$$\Leftrightarrow \begin{array}{l} c_1 \in F^n \setminus \{0\} \\ c_2 \in F^n \setminus \mathrm{span}(c_1) \\ c_2 \in F^n \setminus \mathrm{span}(c_1, c_2) \\ \vdots \\ c_n \in F^n \setminus \mathrm{span}(c_1, \ldots, c_n) \end{array}$$

$\mathrm{span}(c_1, \ldots, c_i) \cong F^i$

$\therefore\ \#\,\mathrm{span}(c_1, \ldots, c_i) = (\#F)^i$

$\therefore\ \#GL_n(F) = ((\#F)^n - 1)((\#F)^n - \#F)((\#F^n) - (\#F)^2) \cdots ((\#F)^n - (\#F)^{n-1})$

Determinants: $\mathrm{Mat}_{n\times n}(F) = \underbrace{F^n \times \cdots \times F^n}_{n}$

$[v_1|v_2| \cdots |v_n] \longleftrightarrow (v_1, \ldots, v_n)\ v_i \in F^n$

v_i's are columns of a matrix

Theorem: For any field (or commutative ring) and any $n \in \mathbb{N}$, there $\exists!$ function $\det : \underbrace{F^n \times \cdots \times F^n}_{n} \to F$ such that

(1) For all $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in F^n$, $\det(v_1, \ldots, v_{i-1}, —, v_{i+1}, \ldots, v_n) : F^n \to F$ is an F linear

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transposition.

⓶ $\det(v_1 m \ldots, v_n) = 0$ if $v_i = v_{i+1}$ for some $1 \le i \le n-1$

⓷ $\det(e_1, e_2, \ldots, e_n) = 1, e_i = i \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Moreover, $\det([a_{ij}]) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdot \cdots \cdot a_{\sigma(n),n}$

Ex $\det(v_1, v_2, \ldots, v_n) \overset{?}{=} \det(v_1 + \lambda v_2, v_2, \ldots, v_n) \overset{①}{=} \begin{matrix} \det(v_1, v_2, \ldots, v_n) \\ + \\ \lambda \det(v_2, v_2, \ldots, v_n) \end{matrix} \overset{②}{=} \det(v_1 n \ldots, v_n)$

Lemma If det satisfies ①—③, then

ⓐ $\det(v_1, v_2, \ldots, v_n) = \text{sign}(\sigma) \cdot \det(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})\ \forall \sigma \in S_n$

ⓑ $\det(v_1, \ldots, v_n) = 0$ if $v_i = v_j$ for some $i \neq j$.

Pf ⓐ σ is a product of adjacent transpositions and sign : $S_n \to \{\pm 1\}$ is a group homomorphism, $\Rightarrow$ it suffices to assume $r = (i\ i+1)$.

For simplicity, let $i = 1$. $0 \overset{②}{=} \det(v_1 + v_2, v_1 + v_2, v_3, v_4, \ldots, v_n) \overset{①}{=} \begin{matrix} \det(v_1, v_1 + v_2, v_3, \ldots, v_n) \\ + \\ \det(v_2, v_1 + v_2, v_3, \ldots, v_n) \end{matrix} \overset{①}{=}$

$$\begin{matrix} \det(v_1, v_1, v_2, \ldots) + \det(v_1, v_2, v_3, \ldots) \\ + \det(v_2, v_1, v_3, \ldots) + \det(v_2, v_2, v_3, \ldots) \end{matrix} \overset{②}{=} \begin{matrix} \det(v_1, v_2, v_3, \ldots, n) \\ + \\ \det(v_2, v_1, v_3, \ldots, v_n) \end{matrix} = 0$$

ⓑ Follows from ⓐ and ②.

Proof Existence: Check the formula given satisfies ①—③, on your own time.

Uniqueness: We show if det satisfies ①—③, then det is given by the formula.

Given $v_1, v_2, \ldots, v_n, v_i = a_{1i} e_1 + a_{2i} e_2 + \cdots + a_{ni} e_n$

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$$\begin{aligned}
\det(v_1, v_2, \ldots, v_n) &\overset{①}{=} \sum_{j_1=1}^{n} a_{j_1,1} \det(e_{j_1}, v_2, v_3, \ldots, v_n) \\
&\overset{①}{=} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} a_{j_1,1} a_{j_2,2} \det(e_{j_1}, e_{j_2}, v_3, v_4, \ldots, v_n) \\
&\overset{①}{=} \cdots \overset{①}{=} \sum_{j_1,\ldots,j_n=1}^{n} a_{j_1,1} a_{j_2,2} a_{j_3,3} \cdots a_{j_n,n} \det(e_{j_1}, e_{j_2}, e_{j_3}, \ldots, e_{j_n}) \\
&\underset{\text{Lemma}ⓑ}{=} \sum_{\sigma \in S_n} a_{\sigma(1),1} a_{\sigma(2),2}, a_{\sigma(3),3} \cdots a_{\sigma(n),n} \det(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, \ldots, e_{\sigma(n)}) \\
&\underset{\text{Lemma}ⓐ}{=} \sum_{\sigma} a_{\sigma(1),1} a_{\sigma(2),2}, a_{\sigma(3),3} \cdots a_{\sigma(n),n} \cdot \operatorname{sign}(\sigma) \cdot \det(e_1, e_2, \ldots, e_n) = *
\end{aligned}$$