MATH 817 Notes

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Prop If $H \trianglelefteq G$, $K \trianglelefteq G$ and $H \cap K = \{e\}$, then the function $\varphi : H \times K \to G$ defined by $\varphi(h,k) = h \cdot k$ is a 1-1 group homomorphism with image $H \cdot K$. In particular, if also $H \cdot K = G$, then $H \times K \cong G$.

Ex $G = \langle x \rangle, |x| = n, n = m \cdot \ell, \gcd(m, \ell) = 1$. Let $a = x^m, b = x^\ell$. $\langle a \rangle \trianglelefteq G, \langle b \rangle \trianglelefteq G, \langle a \rangle \cap \langle b \rangle = \{e\}$, since $\gcd(m, \ell) = 1$.

$\therefore\ \varphi : \langle a \rangle \times \langle b \rangle \to G$ is a 1-1 group homomorphism.

$\#(\langle a \rangle \times \langle b \rangle) = n \therefore\ \varphi$ is onto and $G \cong \langle a \rangle \times \langle b \rangle$

ex $\mathbb{Z}/60 \cong \mathbb{Z}/15 \times \mathbb{Z}/4 \cong \underbrace{\mathbb{Z}/3 \times \mathbb{Z}/5}_{\cong \mathbb{Z}/12 \times \mathbb{Z}/5} \times \mathbb{Z}/4 \cong \mathbb{Z}/3 \times \mathbb{Z}/20$

Chinese Remainder Theorem: $\mathbb{Z}/n = \mathbb{Z}/m \times \mathbb{Z}/\ell$ if $n = m\ell, \gcd(m, \ell) = 1$

Pf: Say $\varphi(h_1, k_1) = \varphi(h_2, k_2)$. So, $h_1 k_1 = h_2 k_2 \Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = \{e\}$.

$\therefore\ h_1 = h_2$ and $k_1 = k_2$

$\therefore\ \varphi$ is 1-1 [I've only used $H \cap K = \{e\}$, $H \leq G$ and $K \leq G$ only.]

To show φ is a group homomorphism, first we show $h \cdot k = k \cdot h, \forall h \in H, k \in K$.

Consider $[h, k] = hkh^{-1}k^{-1} \in H$ since $H \trianglelefteq G$ and $[h, k] \in K$ since $K \trianglelefteq G$.

$\therefore\ [h, k] = e$ and so $hk = kh$.

$\varphi((h_1, k_1) \cdot (h_2, k_2)) = \varphi((h_1 h_2, k_1 k_2)) = h_1 h_2 k_1 k_2$ and $\varphi((h_1, k_1)) \cdot \varphi((h_2, k_2)) = h_1 k_1 \cdot h_2 k_2 = \varphi((h_1, k_1) \cdot (h_2, k_2))$ since $k_1 h_2 = h_2 k_1$. $\square$

Note If $H \trianglelefteq G, K \trianglelefteq G, H \cap K = \{e\}, HK = G$, then $G/H \underset{kH \leftarrow\!\shortmid k}{\xleftarrow{\cong}} K$ and $G/K \underset{hK \leftarrow\!\shortmid h}{\xleftarrow{\cong}} H$.

Moreover, using these isomorphisms,

$$G \underset{g \mapsto (gK, gH)}{\xrightarrow{\cong}} G/K \times G/H \cong H \times K$$

$3 \cdot 5 = 15$

$$\underset{|x|=15}{\langle x \rangle} = \mathbb{Z}/15 \leftrightarrows \overset{\langle x^3 \rangle}{\overset{\|}{\mathbb{Z}/3}} \times \overset{\langle x^5 \rangle}{\overset{\|}{\mathbb{Z}/5}}$$

$$\begin{array}{ccc} \bar{i} & \mapsto & (\bar{\bar{i}}, \bar{\bar{\bar{i}}}) \quad \text{is 1-1, onto} \\ 11 & & (\bar{\bar{1}}, \bar{\bar{\bar{3}}}) \\ x^{11} & \leftarrow\!\shortmid & x^5, x^6 \end{array}$$

Ex $G = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R} \right\} \leq GL_2(\mathbb{R})$ ↖ you check

I claim $G \cong D \times U$. Let $D = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{R} \setminus \{0\} \right\}$, $U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{R} \right\}$ (U = unipotent)

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$D \leq G, U \leq G$ are easy to check

$D \trianglelefteq G$ since $D \subseteq Z(G)$.

$$D \cdot U = G : \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \stackrel{?}{=} \underset{\substack{\uparrow \\ D}}{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}} \underset{\substack{\uparrow \\ U}}{\begin{pmatrix} 1 & y/x \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

$U \subseteq N_G(U)$

<u>Claim</u> $\underset{D \subseteq Z(G)}{D \subseteq \overset{}{\underset{\Uparrow}{N_G}}(U)} : \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}$

$D \cap U = \{e_G\}$

$\therefore\ G \cong D \times U$ by the prop

<u>*Note*</u>$D \cong \mathbb{R}^\times$

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y + y' \\ 0 & 1 \end{pmatrix}$$

Suppose $H \trianglelefteq G$, $K \leq G$ (not necessarily normal), $H \cap K = \{e\}$, $H \cdot K = G$. Then the proof of the proposition sow:

$$\varphi : H \times K \to G$$

is 1-1 and onto.

But it need not be a group homomorphism:

$\varphi((h_1, k_2) \cdot (h_2, k_2)) = h_1(h_2)k_1k_2$

$$\varphi((h_1, k_1)) \cdot \varphi((h_2, k_2)) = h_1k_1h_2k_2 = \begin{matrix} \varphi(h_1k_1h_2k_1^{-1}, k_1k_2) \\ \| \\ h_1(k_1h_2k_1^{-1})k_1k_2 \end{matrix}$$

φ is a group homomorphism $\Leftrightarrow k \subseteq N_G(H) \Rightarrow K \trianglelefteq G$

Idea! Make φ into a group homomorphism by redefining multiplication on $H \times K$ to be

$$(h_1, k_1) * (h_2, k_2) := (h_1k_1h_2k_1^{-1}, k_1k_2) \subseteq H \times K$$

Say K is a group that acts on another group H via automorphisms; i.e., there is a group homomorphism $\rho : K \to \mathrm{Aut}(H)$.

Then we define a group $H \rtimes_\rho K = H \rtimes K$ as follows:

The underlying set is $H \times K$

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1h_2^{k_1}, k_1k_2), \text{where } h_2^{k_1} := \rho(k_1)(h_2).$$