# MATH 817 Notes

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Def Quotient Spaces

$W$ is a subspace of $V \Rightarrow$ the quotient group ${}^{V}\!/_{W}$ is also a vector space via $\lambda|v+W| = \lambda v + W$.

Def $V = F$-vector space

$\dim V = \dim_F V = \#B$ where $B$ is any basis

Prop If $W$ is a subspace of $V$, $\dim V = \dim W + \dim({}^{V}\!/_{W})$.

Pf: Pick a basis $B$ of $W$ and a basis $\overline{C}$ of ${}^{V}\!/_{W}$. For each element of $\overline{C}$ pick an element in $V$ that represents that coset, and let $C \subseteq V$ be the set of all these choices.

So, $\overline{C} = \{v + W \mid v \in C\}$ and $v + W \neq v' + W$ for all $v, v' \in C$ with $v \neq v'$.

Claim: $B \cap C = \varnothing$ and $B \cup C$ is a basis of $V$. Granting this,

$$\begin{aligned} \dim V &= \#(B \cup C) \\ &= \#B + \#C \\ &= \#B + \#\overline{C} = \dim W + \dim {}^{V}\!/_{W} \end{aligned}$$

pf of claim

- If $c \in C \cap B$, then $c + W \in \overline{c}$ aand $c + W = 0_{V/W} \Rightarrow\Leftarrow$
- $\operatorname{Span}(B \cup C) = V$ :

  Pick $v \in V$.
  $$\begin{aligned} v + W &\overset{\text{since } \overline{C} \text{ spans } V/W}{=} \sum_{c \in C} \lambda_c (c + W) \\ &= \left( \sum_{c \in C} \lambda_c \cdot c \right) + W \end{aligned}$$
  $\therefore\ v - \sum_c \lambda_c \cdot c \in W$

  $\therefore\ v - \sum_c \lambda_c \cdot c = \sum_{b \in B} \mu_b \cdot b$

  $\therefore\ v \in \operatorname{Span}(B \cup C)$.
- linearly independent: Say $\sum_{b \in B} \mu_b \cdot b + \sum_{c \in C} \lambda_c \cdot c = 0$. Then since $B \subseteq W$, we set

  $$0 + \sum_C \lambda_c (c + W) = 0_{V/W}$$

  since $\overline{C}$ is linerally independent, $\lambda_c = 0\ \forall c \in C$.

  Then $\sum_{b \in B} \mu_b \cdot b = 0 \Rightarrow \mu_b = 0\ \forall b \in B$, since $B$ is linearly independent. $\square$

Prop Let $\varphi : V \to W$ be a linear transformation of vector spaces $V$ and $W$.

(1) $\ker \varphi$ is a subspace of $V$.

(2) $\operatorname{im} \varphi$ is a subspace of $W$

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1st Isom. Thm. (3) ${}^{V}\!/_{\ker\varphi} \cong \operatorname{im}\varphi$, as vector spaces, via $v+\ker\varphi \mapsto \varphi(v)$.

Rank-Nullity Theorem (4) $\dim V = \dim(\ker\varphi) + \dim(\operatorname{im}\varphi)$.

(5) If $\dim V < \infty$ and $\dim W < \infty$, $\exists$ bases $B = (v_1, \dots, v_n)$ and $C = (w_1, \dots, w_m)$ of $V+W$ such that

$$M_B^C(\varphi) = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}, I_r = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{r\times r}, \text{ some } r.$$

(6) Given an $m \times n$ matrix $A$, $A$ is equivalent to

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}, \text{ some } r.$$

Pf: (1) and (2) are easy

(3) We know such an isomorphism of groups exists. Now observe $\varphi(\lambda v) = \lambda\varphi(v) \Rightarrow$ this isomorphism is also $f$-linear.

(4) Follows from 3 and previous result.

(5) Pick a basis $(w_1, \dots, w_r)$ of $\operatorname{im}\varphi$. For each $i$, pick $v_i \in V$ such that $\varphi(v_i) = w_i$. Pick a basis $v_{r+1}, v_{r+2}, \dots, v_\ell$ of $\ker\varphi$. Then the proof of the previous proposition shows that

$$B = (v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_{ell})$$

form a basis of $V$. (Note $\ell = \dim V = n$)

Extend $w_1, \dots, w_r$ to a basis $C = (w_1, w_2, \dots, w_r, w_{r+1}, \dots, w_m)$ of $W$.

Observe $\varphi(v_1) = \begin{cases} w_i & i \le r \\ 0 & i > r \end{cases}$.

The equation follows. □

(6) Apply (5) to $T_A : F^n \to F^m$.

Cor $\varphi : V \to W$ linear transformation, $\dim V = \dim W < \infty$. The following are equivalent:

(1) $\varphi$ is an isomorphism

(2) $\varphi$ is 1-1

(3) $\varphi$ is onto

(4) $\forall$ bases of $V$, $\varphi(B)$ is a basis of $W$

(1) $\Rightarrow$ (4) clear

(4) $\Rightarrow$ (3) clear, since $\operatorname{im}\varphi \supseteq \operatorname{Span}(\varphi(B)) = W$

(3) $\Rightarrow$ (2) By Proposition (4), $\dim(\ker\varphi) = 0 \Rightarrow \ker\varphi = 0 \Rightarrow \varphi$ is 1-1

(2) $\Rightarrow$ (1) By Proposition (4), $\dim(\operatorname{im}\varphi) = \dim V = \dim W$. $\boxed{\therefore\ \operatorname{im}\varphi = W}$